Dr Oliver Mathematics
Mathematics
Logarithms Part 2
Past Examination Questions

This booklet consists of 61 questions across a variety of examination topics.
The total number of marks available is 475.

1. Differentiate with respect to x:

   (a) $[x + \ln(2x)]^3$, (3)

   Solution

   $$\frac{\mathrm{d}}{\mathrm{d}x}([x + \ln(2x)]^3) = \underline{\underline{3[x + \ln(2x)]^2 \left(1 + \frac{1}{x}\right)}}.$$

   $$\mathrm{f}(x) = 3\mathrm{e}^x - \tfrac{1}{2}\ln x - 2,\ x > 2.$$

   (b) $3\mathrm{e}^x - \frac{1}{2}\ln x - 2$, $x > 2$. (3)

   Solution

   $$\frac{\mathrm{d}}{\mathrm{d}x}(3\mathrm{e}^x - \tfrac{1}{2}\ln x - 2) = \underline{\underline{3\mathrm{e}^x - \frac{1}{2x}}}.$$

2. A particular species of orchid is being studied. The population p at time t years after the study started is assumed to be

   $$p = \frac{2800a\mathrm{e}^{0.2t}}{1 + a\mathrm{e}^{0.2t}},$$

   where a is a constant. Given that there were 300 orchids when the study started,

   (a) show that $a = 0.12$, (3)

   Solution

   $$\begin{aligned} 300 = \frac{2800a}{1+a} &\Rightarrow 300(1 + a) = 2800a \\ &\Rightarrow 300 + 300a = 2800a \\ &\Rightarrow 300 = 2500a \\ &\Rightarrow \underline{\underline{a = 0.12}}. \end{aligned}$$

(b) use the equation with $a = 0.12$ to predict the number of years before the population of orchids reaches 1850. (4)

Solution

$$\begin{aligned}1850 = \frac{336\mathrm{e}^{0.2t}}{1+0.12\mathrm{e}^{0.2t}} &\Rightarrow 1850(1+0.12\mathrm{e}^{0.2t}) = 336\mathrm{e}^{0.2t}\\ &\Rightarrow 1850 + 222\mathrm{e}^{0.2t} = 336\mathrm{e}^{0.2t}\\ &\Rightarrow 1850 = 114\mathrm{e}^{0.2t}\\ &\Rightarrow \mathrm{e}^{0.2t} = \tfrac{925}{57}\\ &\Rightarrow 0.2t = \ln\tfrac{925}{57}\\ &\Rightarrow t = 5\ln\tfrac{925}{57}\\ &\Rightarrow t = 13.933\,712\,35 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{t = 14 \text{ (nearest integer)}}}.\end{aligned}$$

(c) Show that $p = \dfrac{336}{0.12+\mathrm{e}^{-0.2t}}$. (1)

Solution

$$p = \frac{336\mathrm{e}^{0.2t}}{1+0.12\mathrm{e}^{0.2t}} = \underline{\underline{\frac{336}{0.12+\mathrm{e}^{-0.2t}}}}.$$

(d) Hence show the population cannot exceed 2800. (2)

Solution

At $t \to \infty$,

$$\frac{336}{0.12+\mathrm{e}^{-0.2t}} \to \frac{336}{0.12} = \underline{\underline{2800}}.$$

3. The point P lies on the curve with equation $y = \ln(\frac{1}{3}x)$. The x-coordinate of P is 3. Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x},\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3} = \frac{1}{3} \Rightarrow m' = -3.$$

Now, $y = \ln 1 = 0$ and so

$$y = -3(x-3) \Rightarrow \underline{\underline{y = -3x + 9}}.$$

4. Differentiate with respect to x, $x^2 e^{3x+2}$. (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 e^{3x+2}) = 2x \times e^{3x+2} + 3x^2 \times e^{3x+2} = \underline{\underline{x e^{3x+2}(2+3x)}}.$$

5. The functions f and g are defined by

$$\begin{aligned} \mathrm{f} &: x \mapsto 2x + \ln 2,\ x \in \mathbb{R}, \\ \mathrm{g} &: x \mapsto e^{2x},\ x \in \mathbb{R}. \end{aligned}$$

(a) Prove that the composite function gf is (4)

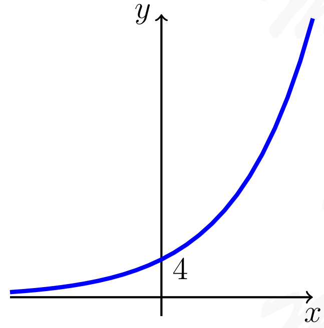
$$\mathrm{g\,f} : x \to 4e^{4x},\ x \in \mathbb{R}.$$

Solution

$$\begin{aligned} \mathrm{g\,f}(x) &= \mathrm{g}(\mathrm{f}(x)) \\ &= \mathrm{g}(2x + \ln 2) \\ &= e^{2(2x+\ln 2)} \\ &= e^{4x+2\ln 2} \\ &= e^{4x+\ln 4} \\ &= e^{\ln 4} e^{4x} \\ &= \underline{\underline{4e^{4x}}}. \end{aligned}$$

(b) Sketch the curve with equation $y = \mathrm{g\,f}(x)$, and show the coordinates of the point where the curve cuts the y-axis. (1)

Solution



(c) Write down the range of g f. (1)

Solution

$\underline{\underline{\mathrm{g\,f}(x) > 0}}$.

(d) Find the value of x for which $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g\,f}(x)] = 3$, giving your answer to 3 significant figures. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g\,f}(x)] = 3 &\Rightarrow 16\mathrm{e}^{4x} = 3\\ &\Rightarrow \mathrm{e}^{4x} = \tfrac{3}{16}\\ &\Rightarrow 4x = \ln\tfrac{3}{16}\\ &\Rightarrow x = \tfrac{1}{4}\ln\tfrac{3}{16}\\ &\Rightarrow x = -0.418\,494\,108\,4 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = -0.418 \text{ (3 sf)}}}\end{aligned}$$

6. Differentiate with respect to x, $\mathrm{e}^{3x} + \ln 2x$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{3x} + \ln 2x) = 3\mathrm{e}^{3x} + \frac{2}{2x} = \underline{\underline{3\mathrm{e}^{3x} + \frac{1}{x}}}.$$

7. A heated metal ball is dropped into a liquid. As the liquid cools, its temperature, T°C,

t minutes after it enters the liquid, is given by

$$T = 400\mathrm{e}^{-0.05t} + 25,\ t \geqslant 0.$$

(a) Find the temperature of the ball as it enter the liquid. (1)

Solution

$400 + 25 = \underline{\underline{425^\circ\mathrm{C}}}$.

(b) Find the value of t for which $T = 300$, giving your answer to 3 significant figures. (4)

Solution

$$\begin{aligned} 400\mathrm{e}^{-0.05t} + 25 = 300 &\Rightarrow 400\mathrm{e}^{-0.05t} = 275 \\ &\Rightarrow \mathrm{e}^{-0.05t} = \tfrac{11}{16} \\ &\Rightarrow -0.05t = \ln \tfrac{11}{16} \\ &\Rightarrow t = -20 \ln \tfrac{11}{16} \\ &\Rightarrow t = 7.493\,868\,989 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 7.49 \text{ (3 sf)}}}. \end{aligned}$$

(c) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures. (3)

Solution

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -20\mathrm{e}^{-0.05t},$$

and we have

$$\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{t=50} = -20\mathrm{e}^{-2.5} = -1.641\,699\,972 \text{ (FCD)} = \underline{\underline{-1.64 \text{ (3 sf)}}}.$$

(d) From the equation for temperature T in terms of t, given above, explain why the temperature of the ball can never fall to 20°C. (1)

Solution

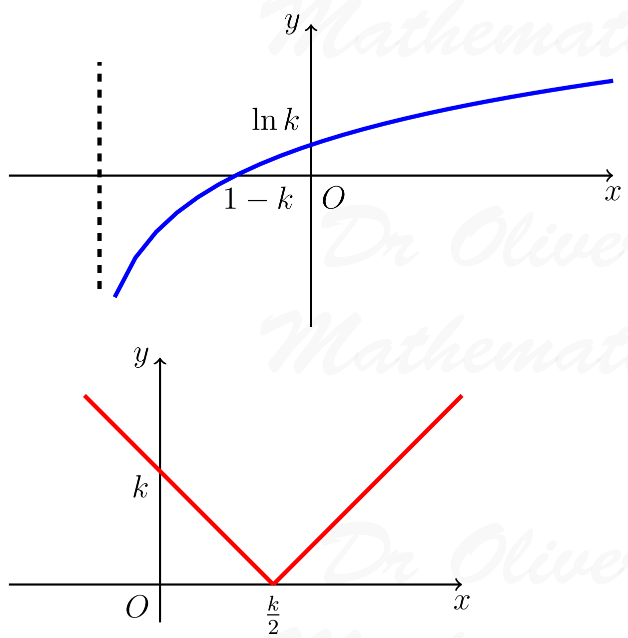
As $t \to \infty$, $\underline{\underline{T \to 25}}$.

8. For the constant k, where $k > 1$, the functions f and g are defined by

$$\mathrm{f} : x \mapsto \ln(x+k),\ x > -k,$$
$$\mathrm{g} : x \mapsto |2x-k|,\ x \in \mathbb{R}.$$

(a) On separate axes, sketch the graph of f and the graph of g. On each sketch, state, in terms of k, the coordinates of the point where the graph meets the coordinate axes. (5)

Solution



(b) Write down the range of f. (1)

Solution

$\mathrm{f}(x) \in \mathbb{R}$.

(c) Find $\mathrm{f}\,\mathrm{g}(\frac{k}{4})$ in terms of k, giving your answer in its simplest form. (2)

Solution

$$\begin{aligned} \mathrm{f\,g}(\tfrac{k}{4}) &= \mathrm{f}(\mathrm{g}(\tfrac{k}{4})) \\ &= \mathrm{f}(|\tfrac{k}{2} - k|) \\ &= \mathrm{f}(\tfrac{k}{2}) \\ &= \ln(k + \tfrac{k}{2}) \\ &= \underline{\underline{\ln \tfrac{3k}{2}}}. \end{aligned}$$

9. Ginven that (5)

$$y = (1 + \mathrm{e}^{2x})^{\frac{3}{2}},$$

find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $\frac{1}{2}\ln 3$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}(1 + \mathrm{e}^{2x})^{\frac{1}{2}} \times 2\mathrm{e}^{2x} = 3\mathrm{e}^{2x}(1 + \mathrm{e}^{2x})^{\frac{1}{2}}$$

and

$$\begin{aligned} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}\ln 3} &= 3\mathrm{e}^{2(\frac{1}{2}\ln 3)}(1 + \mathrm{e}^{2(\frac{1}{2}\ln 3)})^{\frac{1}{2}} \\ &= 3\mathrm{e}^{\ln 3}(1 + \mathrm{e}^{\ln 3})^{\frac{1}{2}} \\ &= 3(3)(1+3)^{\frac{1}{2}} \\ &= \underline{\underline{18}}. \end{aligned}$$

10. The functions f is defined by

$$\mathrm{f} : x \mapsto \ln(4 - 2x),\ x < 2,\ x \in \mathbb{R}.$$

(a) Show that the inverse function of f is defined by (4)

$$\mathrm{f}^{-1} : x \mapsto 2 - \tfrac{1}{2}\mathrm{e}^{x},$$

and write down the domain of f^{-1}.

Solution

$$\begin{aligned} y = \ln(4-2x) &\Rightarrow e^y = 4-2x \\ &\Rightarrow 2x = 4 - e^y \\ &\Rightarrow x = 2 - \tfrac{1}{2}e^y \end{aligned}$$

and so

$$\underline{\underline{f^{-1}(x) = 2 - \tfrac{1}{2}e^x,\ x \in \mathbb{R}.}}$$

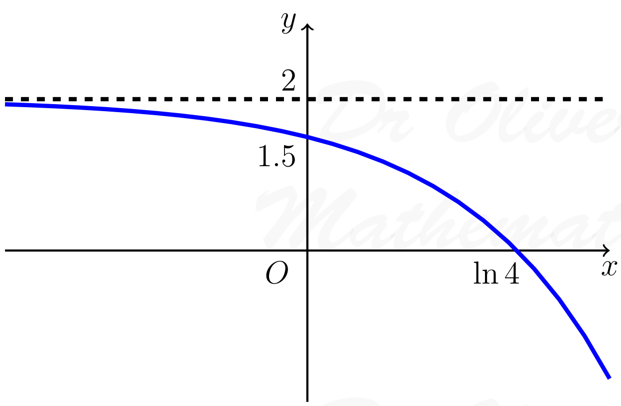
(b) Write down the range of f^{-1}. (1)

Solution

$\underline{\underline{f^{-1}(x) < 2}}$.

(c) Sketch the graph of $y = f^{-1}(x)$. State the coordinates of the points of intersection with the x- and y-axes. (4)

Solution



The graph of $y = x + 2$ crosses the graph of $y = f^{-1}(x)$ at $x = k$. The iterative formula

$$x_{n+1} = -\tfrac{1}{2}e^{x_n},\ x_0 = -0.3,$$

is used to find an approximate value value for k.

(d) Calculate the values of x_1 and x_2, giving your answer to 4 decimal places. (2)

Solution

$x_1 = -0.370\,409\,110\,3$ (FCD) $= \underline{\underline{-0.3704 \text{ (4 dp)}}}$

$x_2 = -0.345\,225\,900\,9$ (FCD) $= \underline{\underline{-0.3452 \text{ (4 dp)}}}$

(e) Find the value of k to 3 decimal places. (2)

Solution

$x_3 = -0.354\,030\,191\,9$ (FCD)
$x_4 = -0.350\,926\,888\,4$ (FCD)
$x_5 = -0.352\,017\,612\,6$ (FCD)
$x_6 = -0.351\,633\,867\,8$ (FCD)
$x_7 = -0.351\,768\,831\,3$ (FCD)

$$x + 2 = 2 - \tfrac{1}{2}\mathrm{e}^x \Rightarrow x + \tfrac{1}{2}\mathrm{e}^x = 0.$$

Put

$$\mathrm{g}(x) = x + \tfrac{1}{2}\mathrm{e}^x.$$

Then
$\mathrm{g}(-0.3525) = -1.03\ldots \times 10^{-3}$
$\mathrm{g}(-0.3515) = 3.15\ldots \times 10^{-4}$
There is a (i) change of sign and (ii) continuity and we have

$$-0.3525 < k < -0.3515,$$

i.e.,

$$\underline{\underline{k = -0.352 \text{ (3 dp)}}}.$$

11. Find the exact solutions to the equations

(a) $\ln x + \ln 3 = \ln 6$, (2)

Solution

$$\begin{aligned}\ln x + \ln 3 = \ln 6 &\Rightarrow \ln 3x = \ln 6\\ &\Rightarrow 3x = 6\\ &\Rightarrow \underline{\underline{x = 2}}.\end{aligned}$$

(b) $\mathrm{e}^x + 3\mathrm{e}^{-x} = 4$. (4)

Solution

$$\begin{aligned}
e^x + 3e^{-x} = 4 &\Rightarrow e^x + 3e^{-x} - 4 = 0\\
&\Rightarrow e^{2x} - 4e^x + 3 = 0\\
&\Rightarrow (e^x - 1)(e^x - 3) = 0\\
&\Rightarrow e^x - 1 = 0 \text{ or } e^x - 3 = 0\\
&\Rightarrow e^x = 1 \text{ or } e^x = 3\\
&\Rightarrow \underline{\underline{x = 0}} \text{ or } \underline{\underline{x = \ln 3}}.
\end{aligned}$$

12. A curve C has equation $y = x^2 e^x$.

(a) Find $\dfrac{dy}{dx}$. (3)

Solution

$$\frac{dy}{dx} = x^2 \times e^x + 2x \times e^x = \underline{\underline{xe^x(x+2)}}.$$

(b) Hence find the coordinates of the turning points of C. (3)

Solution

$$\begin{aligned}
\frac{dy}{dx} = 0 &\Rightarrow xe^x(x+2) = 0\\
&\Rightarrow x = -2 \text{ or } x = 0.
\end{aligned}$$

Now, $x = -2 \Rightarrow y = 4e^{-2}$ and $x = 0 \Rightarrow y = 0$. Hence, the turning points are $\underline{\underline{(-2, 4e^{-2})}}$ and $\underline{\underline{(0, 0)}}$.

(c) Find $\dfrac{d^2y}{dx^2}$. (2)

Solution

$$\begin{aligned}
\frac{dy}{dx} = e^x(x^2 + 2x) &\Rightarrow \frac{d^2y}{dx^2} = e^x \times (x^2 + 2x) + e^x \times (2x + 2)\\
&\Rightarrow \underline{\underline{\frac{d^2y}{dx^2} = e^x(x^2 + 4x + 2)}}.
\end{aligned}$$

(d) Determine the nature of each turning point of the curve C. (2)

Solution

$$x = -2, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2\mathrm{e}^{-2} < 0 \text{ and so this is a } \underline{\underline{\text{maximum}}}$$

and

$$x = 0, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2 > 0 \text{ and so this is a } \underline{\underline{\text{minimum}}}.$$

13. The functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln(2x-1),\ x \in \mathbb{R},\ x > \tfrac{1}{2},$$
$$\mathrm{g}: x \mapsto \frac{2}{x-3},\ x \in \mathbb{R}\, x \neq 3.$$

(a) Find the exact value of $\mathrm{f\,g}(4)$. (2)

Solution

$$\begin{aligned}\mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}\left(\frac{2}{x-3}\right) \\ &= \ln \frac{4}{x-3} - 1\end{aligned}$$

and $\mathrm{f\,g}(4) = \ln(\frac{4}{4-3} - 1) = \underline{\underline{\ln 3}}$.

(b) Find the inverse function $\mathrm{f}^{-1}(x)$, stating its domain. (4)

Solution

$$\begin{aligned}y = \ln(2x-1) &\Rightarrow \mathrm{e}^y = 2x-1 \\ &\Rightarrow \mathrm{e}^y + 1 = 2x \\ &\Rightarrow x = \tfrac{1}{2}(\mathrm{e}^y + 1)\end{aligned}$$

and so

$$\mathrm{f}^{-1}(x) = \underline{\underline{\tfrac{1}{2}(\mathrm{e}^x + 1),\ x \in \mathbb{R}}}.$$

14. The amount of a certain type of drug in the bloodstream t hours after it has been taken is given by the formula

$$x = D\mathrm{e}^{-\frac{1}{8}t},$$

where x is the amount of the drug in the bloodstream in milligrams and D is the dose given in milligrams. A dose of 10 mg of the drug is given.

(a) Find the amount of the drug in the bloodstream 5 hours after the dose is given. Give your answer in mg to 3 decimal places. (2)

Solution

$$x = 10e^{-\frac{5}{8}} = 5.352\,614\,285 \text{ (FCD)} = \underline{\underline{5.353 \text{ (3 dp)}}}.$$

A second dose of 10 mg is given after 5 hours.

(b) Show that the amount of the drug in the bloodstream 1 hour after the second dose is 13.549 mg to 3 decimal places. (2)

Solution

$$x = 10e^{-\frac{1}{8}} + 10e^{-\frac{6}{8}} = 13.548\,634\,55 \text{ (FCD)} = \underline{\underline{13.549 \text{ (3 dp)}}}.$$

No more doses of the drug are given. At time T hours after the the second dose is given, the amount of drug in the bloodstream is 3 mg.

(c) Find the value of T. (3)

Solution

$$\begin{aligned} 15.352\ldots e^{-\frac{1}{8}T} = 3 \Rightarrow e^{-\frac{1}{8}T} &= \frac{3}{15.352\ldots} \\ \Rightarrow -\tfrac{1}{8}T &= \ln \frac{3}{15.352\ldots} \\ \Rightarrow T &= -8 \ln \frac{3}{15.352\ldots} \\ \Rightarrow \underline{\underline{T}} &\underline{\underline{= 13.061\,387\,86 \text{ (FCD)}}}. \end{aligned}$$

15. A curve C has equation

$$y = e^{2x} \tan x,\ x \neq (2n+1)\frac{\pi}{2}.$$

(a) Show that the turning points on C occur where $\tan x = -1$. (6)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2\mathrm{e}^{2x}\tan x + \mathrm{e}^{2x}\sec^2 x = 0 \\ &\Rightarrow \mathrm{e}^{2x}(2\tan x + \sec^2 x) = 0 \\ &\Rightarrow 2\tan x + \sec^2 x = 0 \\ &\Rightarrow 2\tan x + (\tan^2 x + 1) = 0 \\ &\Rightarrow (\tan x + 1)^2 = 0 \\ &\Rightarrow \tan x + 1 = 0 \\ &\Rightarrow \underline{\underline{\tan x = -1}}.\end{aligned}$$

(b) Find an equation of the tangent to C at the point where $x = 0$. (2)

Solution

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and

$$y - 0 = 1(x - 0) \Rightarrow \underline{\underline{y = x}}.$$

16.

$$\mathrm{f}(x) = \ln(x + 2) - x + 1,\ x > -2,\ x \in \mathbb{R}.$$

(a) Show that there is a root of $\mathrm{f}(x) = 0$ in the interval $2 < x < 3$. (2)

Solution

$\mathrm{f}(2) = 0.386\ldots$

$\mathrm{f}(3) = -0.390\ldots$

There is a (i) change of sign and (ii) continuity and we have a root in $\underline{\underline{2 < x < 3}}$.

(b) Use the iterative formula (3)

$$x_{n+1} = \ln(x_n + 2) + 1,\ x_0 = 2.5,$$

to calculate the values of x_1, x_2, and x_3, giving your answer to 5 decimal places.

Solution

$x_1 = 2.504\,077\,397$ (FCD) $= \underline{\underline{2.504\,08\ (5\text{ dp})}}$

$x_2 = 2.504\,983\,075$ (FCD) $= \underline{\underline{2.504\,98\ (5\text{ dp})}}$

$x_3 = 2.505\,184\,134$ (FCD) $= \underline{\underline{2.505\,18\ (5\text{ dp})}}$

(c) Show that $x = 2.505$ is a root of $\mathrm{f}(x) = 0$, correct to 3 decimal places. (2)

Solution
$\mathrm{f}(2.5045) = 5.76\ldots \times 10^{-4}$
$\mathrm{f}(2.5055) = -2.01\ldots \times 10^{-4}$
There is a (i) change of sign and (ii) continuity and we have a root in

$$2.5045 < x < 2.5055,$$

i.e., $\underline{\underline{x = 2.505 \text{ (3 dp)}}}$.

17. The radioactive decay of a substance is given by

$$R = 1000\mathrm{e}^{-ct},\ t \geqslant 0,$$

where R is the number of atoms at time t years and c is a positive constant.

(a) Find the number of atoms when the substance started to decay. (1)

Solution
$\underline{\underline{1000}}$.

It takes 5730 years for half of the substance to decay.

(b) Find the value of c to 3 significant figures. (4)

Solution

$$\begin{aligned} 500 = 1000\mathrm{e}^{-5730c} &\Rightarrow \mathrm{e}^{-5730c} = \tfrac{1}{2} \\ &\Rightarrow -5730c = \ln \tfrac{1}{2} \\ &\Rightarrow c = -\tfrac{1}{5730} \ln \tfrac{1}{2} \\ &\Rightarrow c = 1.209\,680\,943 \times 10^{-4} \text{ (FCD)} \\ &\Rightarrow \underline{\underline{c = 1.21 \times 10^{-4} \text{ (3 sf)}}}. \end{aligned}$$

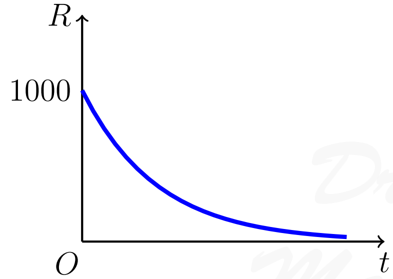
(c) Calculate the number of atoms that will be left when $t = 22\,920$. (2)

Solution

$$t = 22\,920 \Rightarrow R = 1000\mathrm{e}^{-22\,920c} = \underline{\underline{62.5}}.$$

(d) Sketch the graph of R against t. (2)

Solution



18. The point P lies on the curve with equation $y = 4e^{2x+1}$. The y-coordinate of P is 8.

(a) Find, in terms of $\ln 2$, the x-coordinate of P. (2)

Solution

$$\begin{aligned} 4e^{2x+1} = 8 &\Rightarrow e^{2x+1} = 2 \\ &\Rightarrow 2x + 1 = \ln 2 \\ &\Rightarrow 2x = \ln 2 - 1 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}(\ln 2 - 1)}}. \end{aligned}$$

(b) Find the equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found. (4)

Solution

$$\frac{dy}{dx} = 8e^{2x+1} \text{ and } \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}(\ln 2 - 1)} = 16.$$

Now,

$$\begin{aligned} y - 8 = 16[x - \tfrac{1}{2}(\ln 2 - 1)] &\Rightarrow y - 8 = 16x - 8(\ln 2 - 1) \\ &\Rightarrow y - 8 = 16x - 8\ln 2 + 8 \\ &\Rightarrow \underline{\underline{y = 16x - 8\ln 2 + 16}}. \end{aligned}$$

19. Differentiate with respect to x:

(a) $e^{3x}(\sin x + 2\cos x)$, (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^{3x}(\sin x + 2\cos x)] = 3\mathrm{e}^{3x}(\sin x + 2\cos x) + \mathrm{e}^{3x}(\cos x - 2\sin x)$$
$$= \underline{\underline{\mathrm{e}^{3x}(\sin x + 7\cos x)}}.$$

(b) $x^3 \ln(5x + 2)$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^3 \ln(5x+2)] = \underline{\underline{3x^2 \ln(5x+2) + \frac{5x^3}{5x+2}}}.$$

20. The functions f and g are defined by

$$\mathrm{f} : x \mapsto 3x + \ln x,\ x \in \mathbb{R},\ x > 0,$$
$$\mathrm{g} : x \mapsto \mathrm{e}^{x^2},\ x \in \mathbb{R}.$$

(a) Write down the range of g. (1)

Solution

$\underline{\underline{\mathrm{g}(x) \geqslant 1}}$.

(b) Show that the composite function f g is defined by (2)

$$\mathrm{f\,g} : x \mapsto x^2 + 3\mathrm{e}^{x^2},\ x \in \mathbb{R}.$$

Solution

$$\mathrm{f\,g}(x) = \mathrm{f}(\mathrm{g}(x))$$
$$= \mathrm{f}(\mathrm{e}^{x^2})$$
$$= 3\mathrm{e}^{x^2} + \ln \mathrm{e}^{x^2}$$
$$= \underline{\underline{x^2 + 3\mathrm{e}^{x^2}}}.$$

(c) Write down the range of f g. (1)

Solution

$\underline{\underline{\mathrm{f\,g}(x) \geqslant 3}}$.

(d) Solve the equation $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f\,g}(x)] = x(x\mathrm{e}^{x^2} + 2)$. (6)

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f\,g}(x)] = x(x\mathrm{e}^{x^2} + 2) &\Rightarrow 2x + 6x\mathrm{e}^{x^2} = x(x\mathrm{e}^{x^2} + 2) \\ &\Rightarrow 2x + 6x\mathrm{e}^{x^2} = x^2\mathrm{e}^{x^2} + 2x \\ &\Rightarrow 6x\mathrm{e}^{x^2} = x^2\mathrm{e}^{x^2} \\ &\Rightarrow 6x\mathrm{e}^{x^2} - x^2\mathrm{e}^{x^2} = 0 \\ &\Rightarrow x\mathrm{e}^{x^2}(6 - x) = 0 \\ &\Rightarrow \underline{\underline{x = 1}} \text{ or } \underline{\underline{x = 6}}.\end{aligned}$$

21.

$$\mathrm{f}(x) = 3x\mathrm{e}^x - 1.$$

The curve with equation $y = \mathrm{f}(x)$ has a turning point P.

(a) Find the exact coordinates of P. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\mathrm{e}^x + 3x\mathrm{e}^x = 0 \\ &\Rightarrow 3\mathrm{e}^x(1 + x) = 0 \\ &\Rightarrow x = -1\end{aligned}$$

and $y = -3\mathrm{e}^{-1} - 1$. Hence, we have $\underline{\underline{(-1, -3\mathrm{e}^{-1} - 1)}}$.

The equation $\mathrm{f}(x) = 0$ has a root between $x = 0.25$ and $x = 0.3$.

(b) Use the iterative formula (3)

$$x_{n+1} = \tfrac{1}{3}\mathrm{e}^{-x_n},$$

with $x_0 = 0.25$ to calculate the values of x_1, x_2, and x_3, giving your answer to 4 decimal places.

Solution

$x_1 = 0.259\,600\,261$ (FCD) $= \underline{\underline{0.2596 \text{ (4 dp)}}}$

$x_2 = 0.257\,119\,955\,6$ (FCD) $= \underline{\underline{0.2571 \text{ (4 dp)}}}$

$x_3 = 0.257\,758\,483\,2$ (FCD) $= \underline{\underline{0.2576 \text{ (4 dp)}}}$

(c) By choosing a suitable interval, show that a root of $\mathrm{f}(x) = 0$ is $x = 0.2576$, correct to 4 decimal places. (3)

Solution

$\mathrm{f}(0.25755) = -3.79\ldots \times 10^{-4}$
$\mathrm{f}(0.25765) = 1.09\ldots \times 10^{-4}$
There is a (i) change of sign and (ii) continuity and we have a root in

$$0.25755 < x < 0.25765,$$

i.e., $\underline{\underline{x = 0.2576 \text{ (4 dp)}}}$.

22. Rabbits were introduced onto an island. The number of rabbits, P, t years after they were introduced is modelled by the equation

$$P = 80\mathrm{e}^{\frac{1}{5}t},\ t \in \mathbb{R},\ t \geqslant 0.$$

(a) Write down the number of rabbits that were introduced to the island. (1)

Solution

$\underline{\underline{80}}$.

(b) Find the number of years it would take for the number of rabbits to find exceed 1000. (2)

Solution

$$\begin{aligned} 1000 = 80\mathrm{e}^{\frac{1}{5}t} &\Rightarrow \mathrm{e}^{\frac{1}{5}t} = 12.5 \\ &\Rightarrow \tfrac{1}{5}t = \ln 12.5 \\ &\Rightarrow t = 5\ln 12.5 \\ &\Rightarrow \underline{\underline{t = 12.628\,643\,22 \text{ (FCD)}}}. \end{aligned}$$

(c) Find $\dfrac{\mathrm{d}P}{\mathrm{d}t}$. (2)

Solution

$\dfrac{\mathrm{d}P}{\mathrm{d}t} = 16\mathrm{e}^{\frac{1}{5}t}$.

(d) Find P when $\dfrac{\mathrm{d}P}{\mathrm{d}t} = 50$. (3)

Solution

$$\begin{aligned} 50 = 16\mathrm{e}^{\frac{1}{5}t} &\Rightarrow \tfrac{25}{8} = \mathrm{e}^{\frac{1}{5}t} \\ &\Rightarrow \tfrac{1}{5}t = \ln \tfrac{25}{8} \\ &\Rightarrow t = 5\ln \tfrac{25}{8} \\ &\Rightarrow P = 80\mathrm{e}^{\frac{1}{5}\times 5\ln\frac{25}{8}} \\ &\Rightarrow P = 80\mathrm{e}^{\ln\frac{25}{8}} \\ &\Rightarrow P = 80 \times \frac{25}{8} \\ &\Rightarrow \underline{\underline{P = 250}}. \end{aligned}$$

23. Differentiate with respect to x, $\dfrac{\ln(x^2+1)}{x^2+1}$. (4)

Solution

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln(x^2+1)}{x^2+1}\right) &= \frac{(x^2+1)\times\frac{2x}{x^2+1} - 2x\times\ln(x^2+1)}{(x^2+1)^2} \\ &= \underline{\underline{\frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2}}}. \end{aligned}$$

24. Given that

$$\mathrm{f}(x) = \mathrm{e}^{2x} - k,\ x \in \mathbb{R},$$

(a) state the range of f, (1)

Solution

$\underline{\underline{f(x) > -k}}$.

(b) find $f^{-1}(x)$, (3)

Solution

$$\begin{aligned} y = e^{2x} - k &\Rightarrow y + k = e^{2x} \\ &\Rightarrow \ln(y+k) = 2x \\ &\Rightarrow \tfrac{1}{2}\ln(y+k) = x \end{aligned}$$

and hence

$$\underline{\underline{f^{-1}(x) = \tfrac{1}{2}\ln(x+k)}}.$$

(c) write down the domain of f^{-1}. (1)

Solution

$\underline{\underline{f^{-1}(x) > -k}}$.

25. The function g is defined by

$$g(x) = \frac{e^x - 3}{e^x - 2},\ x \in \mathbb{R},\ x \neq \ln 2.$$

(a) Differentiate $g(x)$ to show that (3)

$$g'(x) = \frac{e^x}{(e^x - 2)^2}.$$

Solution

$$\begin{aligned} g(x) = \frac{e^x - 3}{e^x - 2} &\Rightarrow g'(x) = \frac{e^x(e^x - 2) - e^x(e^x - 3)}{(e^x - 2)^2} \\ &\Rightarrow \underline{\underline{g'(x) = \frac{e^x}{(e^x - 2)^2}}}. \end{aligned}$$

(b) Find the exact values of x for which $g'(x) = 1$. (4)

Solution

$$\begin{aligned} g'(x) = 1 &\Rightarrow \frac{e^x}{(e^x - 2)^2} = 1 \\ &\Rightarrow e^x = (e^x - 2)^2 \\ &\Rightarrow e^x = e^{2x} - 4e^x + 4 \\ &\Rightarrow e^{2x} - 5e^x + 4 = 0 \\ &\Rightarrow (e^x - 1)(e^x - 4) = 0 \\ &\Rightarrow e^x - 1 = 0 \text{ or } e^x - 4 = 0 \\ &\Rightarrow \underline{\underline{x = 0}} \text{ or } \underline{\underline{x = \ln 4}}. \end{aligned}$$

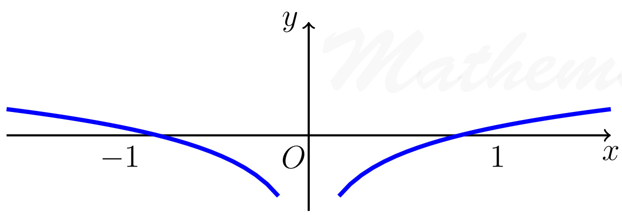
26. Given that $y = \dfrac{\ln(x^2 + 1)}{x}$, find $\dfrac{dy}{dx}$. (4)

Solution

$$\begin{aligned} \frac{d}{dx}\left(\frac{\ln(x^2+1)}{x}\right) &= \frac{x \times \frac{2x}{x^2+1} - 1 \times \ln(x^2+1)}{x^2} \\ &= \underline{\underline{\frac{2x^2 - (x^2+1)\ln(x^2+1)}{x^2(x^2+1)}}}. \end{aligned}$$

27. Sketch the graph of $y = \ln|x|$, stating the coordinates of any points of intersection with the axes. (3)

Solution



28. (a) By writing $\sec x$ as $\dfrac{1}{\cos x}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos x}\right)\\ &= \frac{0\times\cos x - 1\times(-\sin x)}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x}\times\frac{\sin x}{\cos x}\\ &= \underline{\underline{\sec x\tan x}}.\end{aligned}$$

Given that $y = \mathrm{e}^{2x}\sec 3x$,

(b) find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (4)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x}\sec 3x + \mathrm{e}^{2x}\sec 3x\tan 3x = \underline{\underline{\mathrm{e}^{2x}\sec 3x(2+3\tan 3x)}}.$$

The curve with equation $y = \mathrm{e}^{2x}\sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b).

(c) Find the values of the constants a and b, giving your answers to 3 significant figures. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \mathrm{e}^{2x}\sec 3x(2+3\tan 3x) = 0\\ &\Rightarrow 2+3\tan 3x = 0\\ &\Rightarrow \tan 3x = -\tfrac{2}{3}\\ &\Rightarrow 3x = -0.588\,002\,603\,5 \text{ (FCD)}\\ &\Rightarrow x = -0.196\,000\,867\,8 \text{ (FCD)} = \underline{\underline{-0.196 \text{ (3 sf)}}}\\ &\Rightarrow y = 0.812\,093\,867\,1 \text{ (FCD)} = \underline{\underline{0.812 \text{ (3 sf)}}}.\end{aligned}$$

29. (a) Find the exact solutions to the equations

(i) $\ln(3x-7) = 5$, (3)

Solution

$$\begin{aligned}\ln(3x-7) = 5 &\Rightarrow 3x - 7 = \mathrm{e}^5 \\ &\Rightarrow 3x = \mathrm{e}^5 + 7 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{3}(\mathrm{e}^5 + 7)}}.\end{aligned}$$

(ii) $3^x \mathrm{e}^{7x+2} = 15$. (5)

Solution

$$\begin{aligned}3^x \mathrm{e}^{7x+2} = 15 &\Rightarrow \ln(3^x \mathrm{e}^{7x+2}) = \ln 15 \\ &\Rightarrow \ln 3^x + \ln \mathrm{e}^{7x+2} = \ln 15 \\ &\Rightarrow x \ln 3 + 7x + 2 = \ln 15 \\ &\Rightarrow x(\ln 3 + 7) = \ln 15 - 2 \\ &\Rightarrow \underline{\underline{x = \frac{\ln 15 - 2}{\ln 3 + 7}}}.\end{aligned}$$

(b) The functions f and g are defined by

$$\begin{aligned}\mathrm{f} &: x \mapsto \mathrm{e}^{2x} + 3,\ x \in \mathbb{R}, \\ \mathrm{g} &: x \mapsto \ln(x-1),\ x \in \mathbb{R},\ x > 1.\end{aligned}$$

(i) Find f^{-1} and state its domain. (4)

Solution

$$\begin{aligned}y = \mathrm{e}^{2x} + 3 &\Rightarrow y - 3 = \mathrm{e}^{2x} \\ &\Rightarrow \ln(y-3) = 2x \\ &\Rightarrow \tfrac{1}{2}\ln(y-3) = x\end{aligned}$$

and so we have

$$\underline{\underline{\mathrm{f}^{-1}(x) = \tfrac{1}{2}\ln(x-3),\ x > 3}}.$$

(ii) Find f g and state its range. (3)

Solution

$$\begin{aligned} \mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(\ln(x-1)) \\ &= \mathrm{e}^{2\ln(x-1)} + 3 \\ &= \mathrm{e}^{\ln(x-1)^2} + 3 \\ &= (x-1)^2 + 3 \\ &= \underline{\underline{x^2 - 2x + 4,\ \mathrm{f\,g}(x) > 3}}. \end{aligned}$$

30. Figure 1 shows a sketch of the curve C with the equation $y = (2x^2 - 5x + 2)\mathrm{e}^{-x}$.

Figure 1: $y = (2x^2 - 5x + 2)\mathrm{e}^{-x}$

(a) Find the coordinates of the point where C crosses the y-axis. (1)

Solution

$\underline{\underline{(0, 2)}}$.

(b) Show that C crosses the x-axis at $x = 2$ and find the x-coordinate of the other point where C crosses the x-axis. (3)

Solution

$$\begin{aligned} (2x^2 - 5x + 2)\mathrm{e}^{-x} = 0 &\Rightarrow 2x^2 - 5x + 2 = 0 \\ &\Rightarrow (2x-1)(x-2) = 0 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}}} \text{ or } \underline{\underline{x = 2}}. \end{aligned}$$

(c) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x^2 - 5x + 2)\mathrm{e}^{-x} - (4x - 5)\mathrm{e}^{-x} = \underline{\underline{(2x^2 - 9x + 7)\mathrm{e}^{-x}}}.$$

(d) Hence find the exact coordinates of the turning points C. (5)

Solution

$$\begin{aligned}(2x^2 - 9x + 7)\mathrm{e}^{-x} = 0 &\Rightarrow 2x^2 - 9x + 7 = 0\\ &\Rightarrow (2x - 7)(x - 1) = 0\\ &\Rightarrow x = \tfrac{7}{2} \text{ or } \underline{\underline{x = 1}},\end{aligned}$$

and so we have $\underline{\underline{(1, -\mathrm{e}^{-1})}}$ and $\underline{\underline{(\frac{7}{2}, 9\mathrm{e}^{-\frac{7}{2}})}}$.

31. (a) Simplify fully (3)

$$\frac{2x^2 + 9x - 5}{x^2 + 2x - 15}.$$

Solution

$$\begin{aligned}\frac{2x^2 + 9x - 5}{x^2 + 2x - 15} &= \frac{(2x - 1)(x + 5)}{(x - 3)(x + 5)}\\ &= \underline{\underline{\frac{2x - 1}{x - 3}}}.\end{aligned}$$

Given that

$$\ln(2x^2 + 9x - 5) = 1 + \ln(x^2 + 2x - 15),\ x \neq -5,$$

(b) find x in terms of e. (4)

Solution

$$\begin{aligned}
& \ln(2x^2+9x-5) = 1 + \ln(x^2+2x-15) \\
\Rightarrow \quad & \ln(2x^2+9x-5) - \ln(x^2+2x-15) = 1 \\
\Rightarrow \quad & \ln\left(\frac{2x^2+9x-5}{x^2+2x-15}\right) = 1 \\
\Rightarrow \quad & \ln\left(\frac{2x-1}{x-3}\right) = 1 \\
\Rightarrow \quad & \frac{2x-1}{x-3} = \mathrm{e} \\
\Rightarrow \quad & 2x-1 = \mathrm{e}(x-3) \\
\Rightarrow \quad & 2x-1 = \mathrm{e}x - 3\mathrm{e} \\
\Rightarrow \quad & 3\mathrm{e}-1 = x(\mathrm{e}-2) \\
\Rightarrow \quad & \underline{\underline{x = \frac{3\mathrm{e}-1}{\mathrm{e}-2}}}.
\end{aligned}$$

32. Joan brings a cup of hot tea into a room and places the cup on a table. At time t minutes after Joan places the cup on the table, the temperature, θ°C, of the tea is modelled by the equation

$$\theta = 20 + A\mathrm{e}^{-kt},$$

where A and k are positive constants. Given that the initial temperatures of the tea was 90°C,

(a) find the value of A. (2)

Solution

$$90 = 20 + A \Rightarrow \underline{\underline{A = 70}}.$$

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C.

(b) Show that $k = \frac{1}{5}\ln 2$. (3)

Solution

$$55 = 20 + 70e^{-5k} \Rightarrow 35 = 70e^{-5k}$$
$$\Rightarrow \tfrac{1}{2} = e^{-5k}$$
$$\Rightarrow 2 = e^{5k}$$
$$\Rightarrow 5k = \ln 2$$
$$\Rightarrow k = \tfrac{1}{5}\ln 2.$$

(c) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in °C per minute, to 3 decimal places. (3)

Solution

$$\frac{d\theta}{dt} = 70e^{-(\frac{1}{5}\ln 2)t} \times (-\tfrac{1}{5}\ln 2) = -14(\ln 2)e^{-(\frac{1}{5}\ln 2)t}$$

and

$$\left.\frac{d\theta}{dt}\right|_{t=10} = -2.426\,015\,132 \text{ (FCD)}$$

and so the cup of tea is decreasing by 2.426°C (3 dp) per minute.

33. Figure 2 shows a sketch of the curve C with the equation $y = f(x)$, where

$$f(x) = f(x) = (8 - x)\ln x,\ x > 0.$$

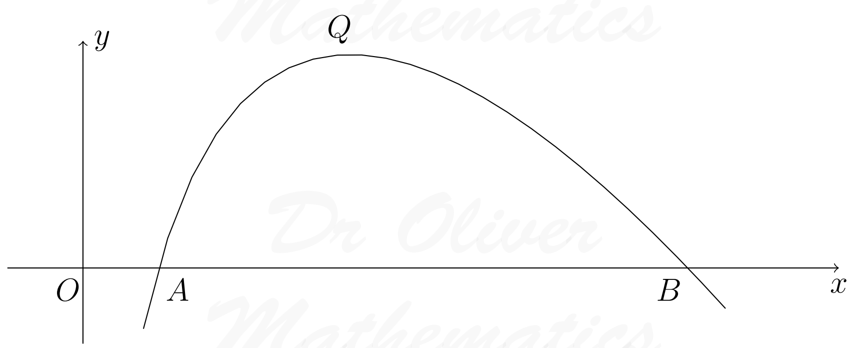


Figure 2: $f(x) = (8 - x)\ln x$

The curve cuts the x-axis at the points A and B and has a maximum turning point at Q.

(a) Write down the coordinates of A and coordinates of B. (2)

Solution

$\underline{\underline{A(1,0)}}$ and $\underline{\underline{B(8,0)}}$.

(b) Find $f'(x)$. (3)

Solution

$$f'(x) = (8-x) \times \frac{1}{x} + (-1) \times \ln x = \underline{\underline{\frac{8-x}{x} - \ln x}}.$$

(c) Show that the x-coordinate of Q lies between 3.5 and 3.6. (2)

Solution

$f'(3.5) = 0.032\ldots$
$f'(3.6) = -0.058\ldots$
There is a (i) change of sign and (ii) continuity and we have a root in

$$\underline{\underline{3.5 < x < 3.6}}.$$

(d) Show that the x-coordinate of Q is the solution of (3)

$$x = \frac{8}{1 + \ln x}.$$

Solution

$$\begin{aligned}\frac{8-x}{x} - \ln x = 0 &\Rightarrow \frac{8-x}{x} = \ln x\\ &\Rightarrow 8 - x = x \ln x\\ &\Rightarrow 8 = x + x\ln x\\ &\Rightarrow 8 = x(1 + \ln x)\\ &\Rightarrow \underline{\underline{x = \frac{8}{1+\ln x}}}.\end{aligned}$$

To find an approximation for the x-coordinate of Q, the iteration formula

$$x_{n+1} = \frac{8}{1 + \ln x_n}$$

is used.

(e) Taking $x_0 = 3.55$, find the values of x_1, x_2, and x_3. Give your answers to 3 decimal places. (3)

Solution

$x_1 = 3.528\,974\,374$ (FCD) $= \underline{\underline{3.529 \text{ (3 dp)}}}$

$x_2 = 3.538\,246\,601\,1$ (FCD) $= \underline{\underline{3.538 \text{ (3 dp)}}}$

$x_3 = 3.534\,144\,722$ (FCD) $= \underline{\underline{3.534 \text{ (3 dp)}}}$

34. Differentiate with respect to x, $x^2 \ln(3x)$. (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \ln(3x)) = 2x \times \ln(3x) + x^2 \times \frac{1}{x} = \underline{\underline{2x \ln(3x) + x}}.$$

35. The area A mm^2, of bacterial culture growing in milk, t hours after midday, is given by

$$A = 20\mathrm{e}^{1.5t}.$$

(a) Write down the area of the culture at midday. (1)

Solution

$\underline{\underline{20}}$.

(b) Find the time at which the area of the culture is twice its area at midday. Give your answer to the nearest minute. (5)

Solution

$$
\begin{aligned}
20\mathrm{e}^{1.5t} = 40 \Rightarrow \mathrm{e}^{1.5t} &= 2 \\
&\Rightarrow 1.5t = \ln 2 \\
&\Rightarrow t = \tfrac{2}{3}\ln 2 \\
&\Rightarrow t = 0.462\,098\,120\,4 \text{ hours (FCD)} \\
&\Rightarrow t = 27.725\,887\,22 \text{ minutes (FCD)} \\
&\Rightarrow \underline{\underline{t = 28 \text{ minutes (nearest minute)}}}.
\end{aligned}
$$

36. The functions f and g are defined by

$$
\begin{aligned}
\mathrm{f} : x &\mapsto \frac{1}{2x-1},\ x \in \mathbb{R},\ x > \tfrac{1}{2}, \\
\mathrm{g} : x &\mapsto \ln(x+1),\ x \in \mathbb{R},\ x > -1.
\end{aligned}
$$

Find the solution of $\mathrm{f}\,\mathrm{g}(x) = \frac{1}{7}$, giving your answer in terms of e.

Solution

$$
\begin{aligned}
\mathrm{f}\,\mathrm{g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\
&= \mathrm{f}(\ln(x+1)) \\
&= \frac{1}{2\ln(x+1)-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2\ln(x+1)-1} = \frac{1}{7} &\Rightarrow 2\ln(x+1) - 1 = 7 \\
&\Rightarrow 2\ln(x+1) = 8 \\
&\Rightarrow \ln(x+1) = 4 \\
&\Rightarrow x + 1 = \mathrm{e}^4 \\
&\Rightarrow \underline{\underline{x = \mathrm{e}^4 - 1}}.
\end{aligned}
$$

37. Figure 3 shows a sketch of the curve C which has equation

$$
y = \mathrm{e}^{x\sqrt{3}} \sin 3x,\ -\tfrac{\pi}{3} \leqslant x \leqslant \tfrac{\pi}{3}.
$$

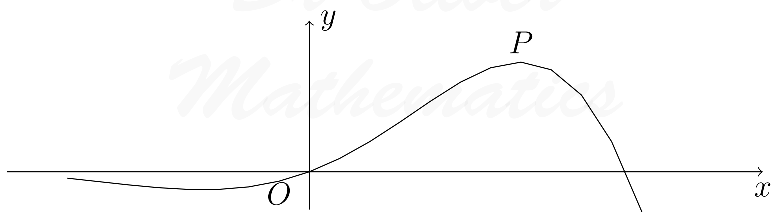


Figure 3: $y = \mathrm{e}^{x\sqrt{3}} \sin 3x$

(a) Find the x-coordinate of the turning point P on C, for which $x > 0$. Give your answer as a multiple of π. (6)

Solution

$$y = \mathrm{e}^{x\sqrt{3}} \sin 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{3}\mathrm{e}^{x\sqrt{3}} \sin 3x + 3\mathrm{e}^{x\sqrt{3}} \cos 3x$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \sqrt{3}\mathrm{e}^{x\sqrt{3}} \sin 3x + 3\mathrm{e}^{x\sqrt{3}} \cos 3x = 0 \\
&\Rightarrow \sqrt{3}\mathrm{e}^{x\sqrt{3}}(\sin 3x + \sqrt{3}\cos 3x) = 0 \\
&\Rightarrow \sin 3x + \sqrt{3}\cos 3x = 0 \\
&\Rightarrow \sin 3x = -\sqrt{3}\cos 3x \\
&\Rightarrow \tan 3x = -\sqrt{3} \\
&\Rightarrow 3x = \tfrac{2\pi}{3} \\
&\Rightarrow \underline{\underline{x = \tfrac{2\pi}{9}}}.
\end{aligned}$$

(b) Find an equation of the normal to C at the point where $x = 0$. (3)

Solution

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \Rightarrow m = -\tfrac{1}{3}$$

and

$$y - 0 = -\tfrac{1}{3}(x - 0) \Rightarrow \underline{\underline{y = -\tfrac{1}{3}x}}.$$

38. The functions f and g are defined by

$$\begin{aligned}
\mathrm{f} &: x \mapsto \mathrm{e}^x + 2,\ x \in \mathbb{R}, \\
\mathrm{g} &: x \mapsto \ln x,\ x \in \mathbb{R},\ x > 0.
\end{aligned}$$

(a) State the range of f. (1)

Solution
$\underline{\underline{f(x) > 2}}$.

(b) Find $\mathrm{f\,g}(x)$, giving your answer in its simplest form. (2)

Solution

$$\begin{aligned} \mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(\ln x) \\ &= \mathrm{e}^{\ln x} + 2 \\ &= \underline{\underline{x + 2}}. \end{aligned}$$

(c) Find the exact value of x for which $\mathrm{f}(2x + 3) = 6$. (4)

Solution

$$\begin{aligned} \mathrm{f}(2x + 3) = 6 &\Rightarrow \mathrm{e}^{2x+3} + 2 = 6 \\ &\Rightarrow \mathrm{e}^{2x+3} = 4 \\ &\Rightarrow 2x + 3 = \ln 4 \\ &\Rightarrow 2x = \ln 4 - 3 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}(\ln 4 - 3)}}. \end{aligned}$$

(d) Find f^{-1}, the inverse function of f, stating its domain. (3)

Solution

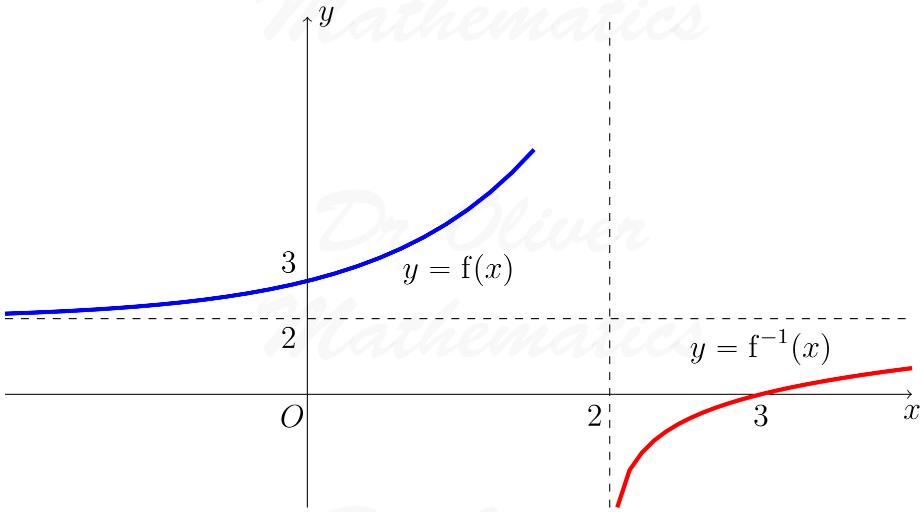
$$\begin{aligned} y = \mathrm{e}^x + 2 &\Rightarrow y - 2 = \mathrm{e}^x \\ &\Rightarrow \ln(y - 2) = x \end{aligned}$$

and so

$$\underline{\underline{\mathrm{f}^{-1}(x) = \ln(x - 2),\ x > 2}}.$$

(e) On the same axes, sketch the curves with equation $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes. (4)

Solution



39. Differentiate with respect to x, $x^{\frac{1}{2}}\ln(3x)$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\frac{1}{2}}\ln(3x)] = \tfrac{1}{2}x^{-\frac{1}{2}} \times \ln(3x) + x^{\frac{1}{2}} \times \frac{1}{x} = \underline{\underline{\tfrac{1}{2}x^{-\frac{1}{2}}[\ln(3x) + 2]}}.$$

40.

$$\mathrm{g}(x) = \mathrm{e}^{x-1} + x - 6.$$

(a) Show that the equation $\mathrm{g}(x) = 0$ can be written as (2)

$$x = \ln(6 - x) + 1,\ x < 6.$$

Solution

$$\begin{aligned} \mathrm{e}^{x-1} + x - 6 = 0 &\Rightarrow \mathrm{e}^{x-1} = 6 - x \\ &\Rightarrow x - 1 = \ln(6 - x) \\ &\Rightarrow \underline{\underline{x = \ln(6 - x) + 1}}. \end{aligned}$$

The root of $g(x) = 0$ is α. The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1,\ x_0 = 2,$$

is used to find an approximate value for α.

(b) Calculate the values of x_1, x_2, and x_3 to 4 decimal places. (3)

Solution

$x_1 = 2.386\,294\,361$ (FCD) $= \underline{\underline{2.3863 \text{ (4 dp)}}}$

$x_2 = 2.284\,733\,739$ (FCD) $= \underline{\underline{2.2847 \text{ (4 dp)}}}$

$x_3 = 2.312\,450\,348$ (FCD) $= \underline{\underline{2.3125 \text{ (4 dp)}}}$

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. (3)

Solution

$g(2.3065) = -2.72\ldots \times 10^{-4}$
$g(2.3075) = 4.41\ldots \times 10^{-3}$
There is a (i) change of sign and (ii) continuity and we have a root in

$$2.3065 < \alpha < 2.3075,$$

i.e.,

$$\underline{\underline{\alpha = 2.307 \text{ (3 dp)}}}.$$

41. Differentiate with respect to x, $x^3 \ln 2x$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^3 \ln 2x] = 3x^2 \times \ln 2x + x^3 \times \frac{1}{x} = \underline{\underline{x^2(3\ln 2x + 1)}}.$$

42. The value of Bob's car can be calculated from the formula

$$V = 17\,000\mathrm{e}^{-0.25t} + 2\,000\mathrm{e}^{-0.5t} + 500,$$

where V is the value of the car in pounds (£) and t is the age in years.

(a) Find the value of the car when $t = 0$. (1)

Solution

$17\,000 + 2\,000 + 500 = \underline{\underline{19\,500}}.$

(b) Calculate the exact value of t when $V = 9\,500$. (4)

Solution

$$
\begin{aligned}
& 9\,500 = 17\,000\mathrm{e}^{-0.25t} + 2\,000\mathrm{e}^{-0.5t} + 500 \\
\Rightarrow\ & 9\,000 = 17\,000\mathrm{e}^{-0.25t} + 2\,000\mathrm{e}^{-0.5t} \\
\Rightarrow\ & 9 = 17\mathrm{e}^{-0.25t} + 2\mathrm{e}^{-0.5t} \\
\Rightarrow\ & 9\mathrm{e}^{0.5t} = 17\mathrm{e}^{0.25t} + 2 \\
\Rightarrow\ & 9\mathrm{e}^{0.5t} - 17\mathrm{e}^{0.25t} - 2 = 0 \\
\Rightarrow\ & (9\mathrm{e}^{0.25t} + 1)(\mathrm{e}^{0.25t} - 2) = 0 \\
\Rightarrow\ & \mathrm{e}^{0.25t} - 2 = 0 \\
\Rightarrow\ & \mathrm{e}^{0.25t} = 2 \\
\Rightarrow\ & 0.25t = \ln 2 \\
\Rightarrow\ & \underline{\underline{t = 4\ln 2}}.
\end{aligned}
$$

(c) Find the rate at which the value of the car is decreasing at the instant when $t = 8$. Give your answer in pounds per year to the nearest pound. (4)

Solution

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -4\,250\mathrm{e}^{-0.25t} - 1\,000\mathrm{e}^{-0.5t}$$

and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{t=8} = -593.490\,592\,6 \text{ (FCD)},$$

a $\underline{\underline{\text{decrease of £593 (nearest pound)}}}$.

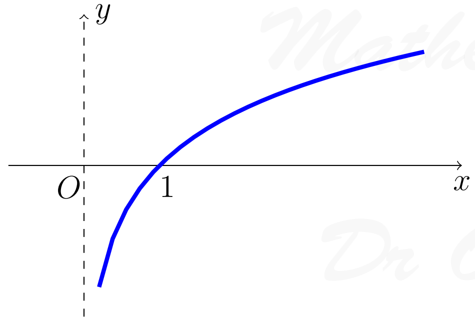
43. Given that

$$\mathrm{f}(x) = \ln x,\ x > 0,$$

sketch on separate axes the graphs of

(a) $y = \mathrm{f}(x)$, (1)

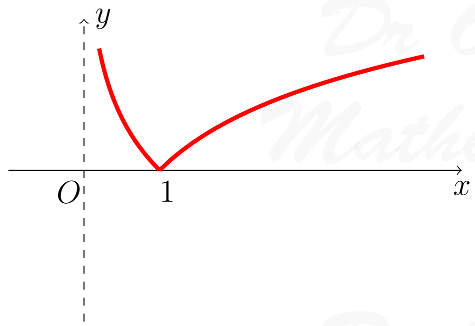
Solution



$\underline{\underline{x = 0}}$

(b) $y = |\,\mathrm{f}(x)|$ (3)

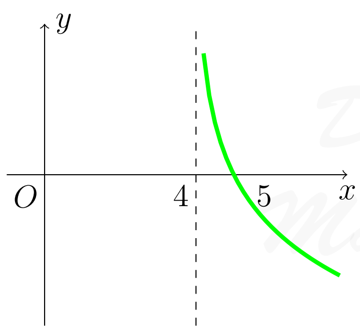
Solution



$\underline{\underline{x = 0}}$

(c) $y = -\,\mathrm{f}(x - 4)$ (3)

Solution



$\underline{\underline{x = 4}}$

Show, in each case, the point where the graph meets or crosses the x-axis. In each case, state the equation of the asymptote.

44.

$$f(x) = 25x^2e^{2x} - 16,\ x \in \mathbb{R}.$$

(a) Using calculus, find the exact coordinates of the turning points on the curve with equation $y = f(x)$. (5)

Solution

$$\begin{aligned} f'(x) = 0 &\Rightarrow 50xe^{2x} + 50x^2e^{2x} = 0 \\ &\Rightarrow 50xe^{2x}(1+x) = 0 \\ &\Rightarrow x(1+x) = 0 \\ &\Rightarrow x = -1 \text{ or } x = 0, \end{aligned}$$

and we have $\underline{\underline{(-1, 25e^{-2} - 16)}}$ and $\underline{\underline{(0, -16)}}$.

(b) Show that the equation $f(x) = 0$ can be written as $x = \pm\frac{4}{5}e^{-x}$. (1)

Solution

$$25x^2e^{2x} - 16 = 0 \Rightarrow x^2 = \tfrac{16}{25}e^{-2x} \Rightarrow \underline{\underline{x = \pm\tfrac{4}{5}e^{-x}}}.$$

The equation $f(x) = 0$ has a root α, where $\alpha = 0.5$ to 1 decimal place.

(c) Starting with $x_0 = 0.5$, use the iteration formula (3)

$$x_{n+1} = \tfrac{4}{5}e^{-x_n}$$

to calculate the values of x_1, x_2, and x_3 to 3 decimal places.

Solution

$x_1 = 0.485\,224\,527\,8$ (FCD) $= \underline{\underline{0.485\ (3\text{ dp})}}$

$x_2 = 0.492\,447\,176\,9$ (FCD) $= \underline{\underline{0.492\ (3\text{ dp})}}$

$x_3 = 0.488\,903\,217\,5$ (FCD) $= \underline{\underline{0.489\ (3\text{ dp})}}$

(d) Give an accurate estimate for α to 2 decimal places, and justify your answer. (2)

Solution

$f(0.485) = -0.487\ldots$
$f(0.495) = 0.485\ldots$

There is a (i) change of sign and (ii) continuity and we have a root in

$$0.485 < \alpha < 0.495,$$

i.e.,

$$\underline{\underline{\alpha = 0.49 \text{ (2 dp)}}}.$$

45. Find algebraically the exact solutions to the equations

(a) $\ln(4-2x)+\ln(9-3x) = 2\ln(x+1)$, $-1 < x < 2$, (5)

Solution

$$\begin{aligned}
& \ln(4-2x)+\ln(9-3x) = 2\ln(x+1) \\
\Rightarrow\ & \ln[(4-2x)(9-3x)] = \ln[(x+1)^2] \\
\Rightarrow\ & (4-2x)(9-3x) = (x+1)^2 \\
\Rightarrow\ & 6x^2 - 30x + 36 = x^2 + 2x + 1 \\
\Rightarrow\ & 5x^2 - 32x + 35 = 0 \\
\Rightarrow\ & (5x-7)(x-5) = 0 \\
\Rightarrow\ & 5x - 7 = 0 \text{ (as } x - 5 = 0 \text{ won't work)} \\
\Rightarrow\ & \underline{\underline{x = 1\tfrac{2}{5}}}.
\end{aligned}$$

(b) $2^x e^{3x+1} = 10$. (5)

Solution

$$\begin{aligned}
2^x e^{3x+1} = 10 &\Rightarrow \ln(2^x e^{3x+1}) = \ln 10 \\
&\Rightarrow \ln 2^x + \ln e^{3x+1} = \ln 10 \\
&\Rightarrow x\ln 2 + 3x + 1 = \ln 10 \\
&\Rightarrow x(\ln 2 + 3) = -1 + \ln 10 \\
&\Rightarrow \underline{\underline{x = \frac{-1+\ln 10}{3+\ln 2}}}.
\end{aligned}$$

Give your answer to (b) in the form $\dfrac{a+\ln b}{c+\ln d}$, where a, b, c, and d are integers.

46. Figure 4 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}.$$

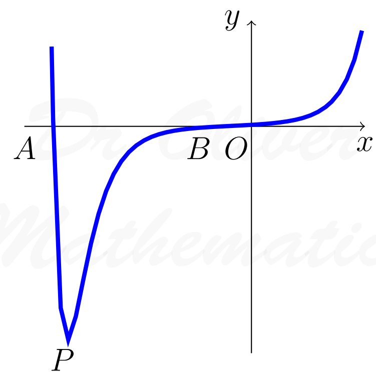


Figure 4: $\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}$

The curve cuts the x-axis at the points A and B.

(a) Calculate the x-coordinate of A and the x-coordinate of B, giving your answers to 3 decimal places. (2)

Solution

$$\begin{aligned}
(x^2 + 3x + 1)\mathrm{e}^{x^2} = 0 &\Rightarrow x^2 + 3x + 1 = 0 \\
&\Rightarrow x^2 + 3x + 2.25 = 1.25 \\
&\Rightarrow (x + 1.5)^2 = 1.25 \\
&\Rightarrow x + 1.5 = \pm\frac{\sqrt{5}}{2} \\
&\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2} \\
&\Rightarrow \underline{\underline{x = -2.618, -0.382 \text{ (3 dp)}}}.
\end{aligned}$$

(b) Find $\mathrm{f}'(x)$. (3)

Solution

$$\mathrm{f}'(x) = 2x(x^2 + 3x + 1) \times \mathrm{e}^{x^2} + (2x + 3) \times \mathrm{e}^{x^2} = \underline{\underline{(2x^3 + 6x^2 + 4x + 3)\mathrm{e}^{x^2}}}.$$

The curve has a minimum turning point at the point P.

(c) Show that the x-coordinate of P is the solution of (3)

$$x = -\frac{3(2x^2+1)}{2(x^2+2)}.$$

Solution

$$\begin{aligned}(2x^3+6x^2+4x+3)\mathrm{e}^{x^2} = 0 &\Rightarrow 2x^3+6x^2+4x+3 = 0 \\ &\Rightarrow 2x(x^2+2) = -3(2x^2+1) \\ &\Rightarrow \underline{\underline{x = -\frac{3(2x^2+1)}{2(x^2+2)}}}.\end{aligned}$$

(d) Use the iteration formula (3)

$$x_{n+1} = -\frac{3(2x_n^2+1)}{2(x_n^2+2)}, \text{ with } x_0 = -2.4,$$

to calculate the values of x_1, x_2, and x_3 to 3 decimal places.

Solution

$x_1 = -2.420\,103\,093$ (FCD) $= \underline{\underline{-2.420 \text{ (3 dp)}}}$

$x_2 = -2.427\,254\,95$ (FCD) $= \underline{\underline{-2.427 \text{ (3 dp)}}}$

$x_3 = -2.429\,771\,016$ (FCD) $= \underline{\underline{-2.430 \text{ (3 dp)}}}$

The x-coordinate of P is α.

(e) By choosing a suitable interval, prove that $\alpha = -2.43$ to 2 decimal places. (2)

Solution

$\mathrm{f}(-2.435) = -15.0\ldots$
$\mathrm{f}(-2.425) = 22.4\ldots$
There is a (i) change of sign and (ii) continuity and we have a root in

$$-2.435 < \alpha < -2.425,$$

i.e.,

$$\underline{\underline{\alpha = -2.43 \text{ (2 dp)}}}.$$

47. The population of a town is being studied. The population P, at time t years from the start of the study, is assumed to be

$$P = \frac{8000}{1 + 7\mathrm{e}^{-kt}},$$

where k is a positive constant. Use the given equation to

(a) find the population at the start of the study, (2)

Solution

$$P = \frac{8000}{1+7} = \underline{\underline{1000}}.$$

(b) find a value for the expected upper limit of the population. (1)

Solution

As $t \to \infty$, $\underline{\underline{P \to 8000}}$.

Given also that the population reaches 2500 at 3 years from the start of the study,

(c) calculate the value of k to 3 decimal places. (5)

Solution

$$\begin{aligned}\frac{8000}{1 + 7\mathrm{e}^{-3k}} = 2500 &\Rightarrow \frac{1}{1 + 7\mathrm{e}^{-3k}} = \tfrac{5}{16}\\ &\Rightarrow \frac{1}{\frac{5}{16}} = 1 + 7\mathrm{e}^{-3k}\\ &\Rightarrow \tfrac{16}{5} = 1 + 7\mathrm{e}^{-3k}\\ &\Rightarrow \tfrac{11}{5} = 7\mathrm{e}^{-3k}\\ &\Rightarrow \tfrac{11}{35} = \mathrm{e}^{-3k}\\ &\Rightarrow -3k = \ln \tfrac{11}{35}\\ &\Rightarrow k = -\tfrac{1}{3}\ln \tfrac{11}{35}\\ &\Rightarrow k = 0.385\,817\,596\,2 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{k = 0.386 \text{ (3 dp)}}}.\end{aligned}$$

Using this value for k,

(d) find the population at 10 years from the start of the study, giving your answer to 3 significant figures. (2)

Solution

$$t = 10 \Rightarrow P = \frac{8000}{1 + 7\mathrm{e}^{-10k}} = 6970.187\,89 \text{ (FCD)} = \underline{\underline{6970 \text{ people (3 sf)}}}.$$

(e) Find, using $\frac{\mathrm{d}P}{\mathrm{d}t}$, the rate at which the population is growing at 10 years from the start of the study. (3)

Solution

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{0 - 56\,000k\mathrm{e}^{-kt}}{(1 + 7\mathrm{e}^{-kt})^2} = \frac{56\,000k\mathrm{e}^{-kt}}{(1 + 7\mathrm{e}^{-kt})^2}$$

and

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t}\right|_{t=10} = \underline{\underline{346.174\,061\,7 \text{ (FCD)}}}.$$

48. Find the exact solutions, in their simplest form, to the equations

(a) $2\ln(2x + 1) - 10 = 0$, (2)

Solution

$$\begin{aligned} 2\ln(2x+1) - 10 = 0 &\Rightarrow 2\ln(2x+1) = 10 \\ &\Rightarrow \ln(2x+1) = 5 \\ &\Rightarrow 2x + 1 = \mathrm{e}^5 \\ &\Rightarrow 2x = \mathrm{e}^5 - 1 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}(\mathrm{e}^5 - 1)}}. \end{aligned}$$

(b) $3^x\mathrm{e}^{4x} = \mathrm{e}^7$. (4)

Solution

$$\begin{aligned} 3^x\mathrm{e}^{4x} = \mathrm{e}^7 &\Rightarrow \ln(3^x\mathrm{e}^{4x}) = \ln\mathrm{e}^7 \\ &\Rightarrow \ln 3^x + \ln\mathrm{e}^{4x} = 7 \\ &\Rightarrow x\ln 3 + 4x = 7 \\ &\Rightarrow x(\ln 3 + 4) = 7 \\ &\Rightarrow \underline{\underline{x = \frac{7}{\ln 3 + 4}}}. \end{aligned}$$

49. A rare species of primrose is being studied. The population, P, of primroses at time t years after the study started is modelled by the equation

$$P = \frac{800\mathrm{e}^{0.1t}}{1 + 3\mathrm{e}^{0.1t}},\ t \geqslant 0,\ t \in \mathbb{R}.$$

(a) Calculate the number of primroses at the start of the study. (2)

Solution

$$P = \frac{800}{1+3} = \frac{800}{4} = \underline{\underline{200}}.$$

(b) Find the exact value of t when $P = 250$, giving your answer in the form $a \ln b$, where a and b are integers. (4)

Solution

$$\begin{aligned}\frac{800\mathrm{e}^{0.1t}}{1 + 3\mathrm{e}^{0.1t}} = 250 &\Rightarrow 800\mathrm{e}^{0.1t} = 250(1 + 3\mathrm{e}^{0.1t}) \\ &\Rightarrow 800\mathrm{e}^{0.1t} = 250 + 750\mathrm{e}^{0.1t} \\ &\Rightarrow 50\mathrm{e}^{0.1t} = 250 \\ &\Rightarrow \mathrm{e}^{0.1t} = 5 \\ &\Rightarrow 0.1t = \ln 5 \\ &\Rightarrow \underline{\underline{t = 10 \ln 5}}.\end{aligned}$$

(c) Find the exact value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ when $t = 10$. Give your answer in its simplest form. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{(1 + 3\mathrm{e}^{0.1t}) \times 80\mathrm{e}^{0.1t} - 0.3\mathrm{e}^{0.1t} \times 800\mathrm{e}^{0.1t}}{(1 + 3\mathrm{e}^{0.1t})^2} \\ &= \frac{80\mathrm{e}^{0.1t}}{(1 + 3\mathrm{e}^{0.1t})^2}\end{aligned}$$

and

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t}\right|_{t=10} = \underline{\underline{\frac{80\mathrm{e}}{(1 + 3\mathrm{e})^2}}}.$$

(d) Explain why the population of primroses can never be 270. (1)

Solution

As $t \to \infty$, $P \to \frac{800}{3} = \underline{\underline{266\frac{2}{3}}}$.

50. A curve C has the equation $y = \mathrm{e}^{4x} + x^4 + 8x + 5$.

(a) Show that the x-coordinate of any turning point of C satisfies the equation (3)

$$x^3 = -2 - \mathrm{e}^{4x}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 4\mathrm{e}^{4x} + 4x^3 + 8 = 0\\ &\Rightarrow 4x^3 = -4\mathrm{e}^{4x} - 8\\ &\Rightarrow \underline{\underline{x^3 = -2 - \mathrm{e}^{4x}}}.\end{aligned}$$

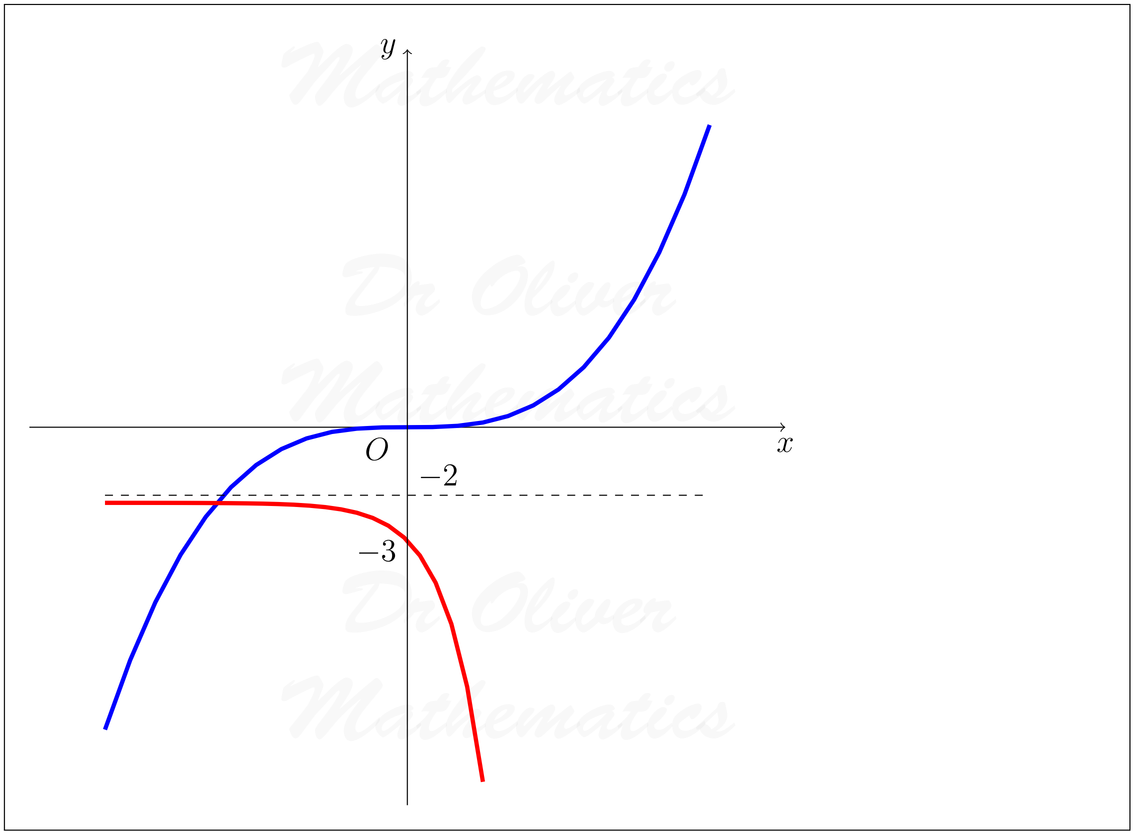
(b) Sketch, on a single diagram, the curves with equations (4)

(i) $y = x^3$,

(ii) $y = -2 - \mathrm{e}^{4x}$.

On your diagram, give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

Solution



(c) Explain how your diagram illustrates that the equation $x^3 = -2 - e^{4x}$ has only one root. (1)

Solution
The two sketches have only one crossing point.

The iteration formula

$$x_{n+1} = (-2 - e^{4x_n})^{\frac{1}{3}},\ x_0 = -1,$$

can be used to find an approximate value for this root.

(d) Calculate the values of x_1 and x_2, giving your answer to 5 decimal places. (2)

Solution
$x_1 = -1.263\,755\,412$ (FCD) $= -1.263\,76$ (5 dp)

$x_2 = -1.261\,258\,763$ (FCD) $= -1.261\,26$ (5 dp)

(e) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve C. (2)

Solution

Let $f(x) = 4e^{4x} + 4x^3 + 8$. Now,
$f(-1.265) = -0.071\ldots$
$f(-1.255) = 0.11\ldots$
There is a (i) change of sign and (ii) continuity and we have a root in

$$-1.265 < \alpha < -1.255,$$

i.e.,

$$\text{the turning point is } \underline{\underline{(-1.26, -2.55) \text{ (2 dp)}}}.$$

51. Given that (5)

$$y = (x^2 + x^3)\ln 2x,$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{e}{2}$, giving your answer in its simplest form.

Solution

$$\frac{dy}{dx} = (x^2 + x^3) \times \frac{1}{x} + (2x + 3x^2) \times \ln 2x = x + x^2 + (2x + 3x^2)\ln 2x$$

and

$$\left.\frac{dy}{dx}\right|_{x=\frac{e}{2}} = \frac{e}{2} + \left(\frac{e}{2}\right)^2 + (e + \tfrac{3}{4}e^2)\ln e$$
$$= \underline{\underline{\tfrac{3}{2}e + e^2}}.$$

52. The functions f is defined by

$$f : x \mapsto e^{2x} + k^2,\ x \in \mathbb{R},$$

where k is a positive constant.

(a) State the range of f. (1)

Solution

$\underline{\underline{f(x) > k^2}}$.

(b) Find f^{-1} and state its domain. (3)

Solution

$$\begin{aligned} y = \mathrm{e}^{2x} + k^2 &\Rightarrow y - k^2 = \mathrm{e}^{2x} \\ &\Rightarrow \ln(y - k^2) = 2x \\ &\Rightarrow \tfrac{1}{2}\ln(y - k^2) = x \end{aligned}$$

and so

$$\underline{\underline{\mathrm{f}^{-1}(x) = \tfrac{1}{2}\ln(x - k^2)\, x > k^2}}.$$

The functions g is defined by

$$\mathrm{g} : x \mapsto \ln(2x),\ x > 0.$$

(c) Solve the equation (4)

$$\mathrm{g}(x) + \mathrm{g}(x^2) + \mathrm{g}(x^3) = 6,$$

giving your answer in its simplest form.

Solution

$$\begin{aligned} \mathrm{g}(x) + \mathrm{g}(x^2) + \mathrm{g}(x^3) = 6 &\Rightarrow \ln(2x) + \ln(2(x^2)) + \ln(2(x^3)) = 6 \\ &\Rightarrow \ln(2x) + \ln(2x^2) + \ln(2x^3) = 6 \\ &\Rightarrow \ln(8x^6) = 6 \\ &\Rightarrow 8x^6 = \mathrm{e}^6 \\ &\Rightarrow x^6 = \tfrac{1}{8}\mathrm{e}^6 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{\sqrt{2}}\mathrm{e}}}. \end{aligned}$$

(d) Find $\mathrm{f\,g}(x)$, giving your answer in its simplest form. (2)

Solution

$$\begin{aligned} \mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(\ln(2x)) \\ &= \mathrm{e}^{2\ln(2x)} + k^2 \\ &= \mathrm{e}^{\ln(4x^2)} + k^2 \\ &= \underline{\underline{4x^2 + k^2}}. \end{aligned}$$

(e) Find, in terms of the constant k, the solution of the equation (2)

$$\mathrm{f}\,\mathrm{g}(x) = 2k^2.$$

Solution

$$4x^2 + k^2 = 2k^2 \Rightarrow 4x^2 = k^2 \Rightarrow x^2 = \tfrac{1}{2}k^2 \Rightarrow \underline{\underline{x = \tfrac{1}{2}k}}$$

and we have $' + \text{ve}'$ since k is a positive constant.

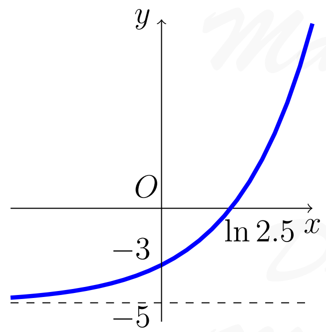
53. Given that

$$\mathrm{f}(x) = 2\mathrm{e}^x - 5,\ x \in \mathbb{R},$$

(a) sketch, on separate diagrams, the curves with equation

(i) $y = \mathrm{f}(x)$,

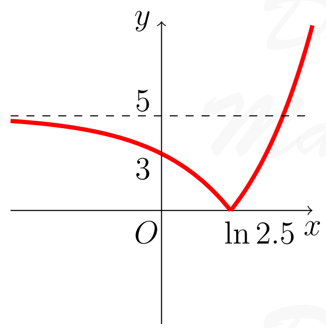
Solution



$\underline{\underline{y = -5}}$

(ii) $y = |\,\mathrm{f}(x)|$.

Solution



$\underline{\underline{y = 5}}$

On each diagram, show the coordinates of each point at which the curve meets or cuts the axes. On each diagram, state the equation of the asymptote.

(b) Deduce the set of values of x for which $\mathrm{f}(x) = |\,\mathrm{f}(x)|$. (1)

> **Solution**
> $\underline{\underline{x \geqslant \ln 2.5}}$.

(c) Find the exact solutions of the equation $|\,\mathrm{f}(x)| = 2$. (2)

> **Solution**
> $\underline{\mathrm{f}(x) = 2}$:
> $$2\mathrm{e}^x - 5 = 2 \Rightarrow \mathrm{e}^x = 3.5 \Rightarrow \underline{\underline{x = \ln 3.5}}.$$
> $\underline{\mathrm{f}(x) = -2}$:
> $$2\mathrm{e}^x - 5 = -2 \Rightarrow \mathrm{e}^x = 1.5 \Rightarrow \underline{\underline{x = \ln 1.5}}.$$

54. Water is being heated in an electric kettle. The temperature, θ°C, of the water t seconds after the kettle is switched on, is modelled by the equation

$$\theta = 120 - 100\mathrm{e}^{-\lambda t},\ 0 \leqslant t \leqslant T.$$

(a) State the value of θ when $t = 0$. (1)

> **Solution**
> $120 - 100 = \underline{\underline{20°\mathrm{C}}}$.

Given that the temperature of the water in the kettle is 70°C when $t = 40$,

(b) find the exact value for λ, giving your answer in the form $\frac{\ln a}{b}$, where a and b are integers. (4)

> **Solution**
> $$\begin{aligned} 120 - 100\mathrm{e}^{-40\lambda} = 70 &\Rightarrow 50 = 100\mathrm{e}^{-40\lambda} \\ &\Rightarrow \tfrac{1}{2} = \mathrm{e}^{-40\lambda} \\ &\Rightarrow \ln\tfrac{1}{2} = -40\lambda \\ &\Rightarrow \lambda = -\frac{\ln\frac{1}{2}}{40} \\ &\Rightarrow \underline{\underline{\lambda = \frac{\ln 2}{40}}}. \end{aligned}$$

When $t = T$, the temperature of the water reaches 100°C and the kettle switches off.

(c) Calculate the value of T to the nearest whole number. (2)

Solution

$$\begin{aligned} 120 - 100e^{-\lambda T} = 100 &\Rightarrow 20 = 100e^{-\lambda T} \\ &\Rightarrow \tfrac{1}{5} = e^{-\lambda T} \\ &\Rightarrow \ln \tfrac{1}{5} = -\lambda T \\ &\Rightarrow T = -\frac{\ln \frac{1}{5}}{\lambda} \\ &\Rightarrow T = 92.877\,123\,8 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{T = 93 \text{ (nearest whole number)}}}. \end{aligned}$$

55. Figure 5 is a sketch showing part of the curve with equation $y = 2^{x+1} - 3$ and part of the line with equation $y = 17 - x$.

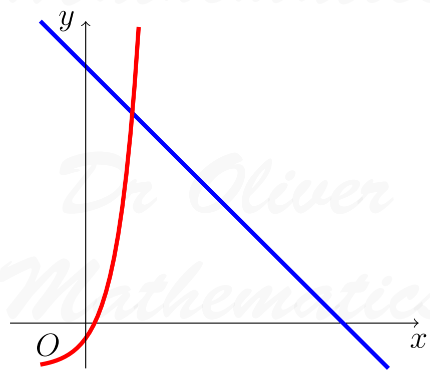


Figure 5: $y = 2^{x+1} - 3$ and $y = 17 - x$

(a) Show that the x-coordinate of A satisfies the equation (3)

$$x = \frac{\ln(20 - x)}{\ln 2} - 1.$$

Solution

$$
\begin{aligned}
2^{x+1} - 3 = 17 - x &\Rightarrow 2^{x+1} = 20 - x \\
&\Rightarrow \ln 2^{x+1} = \ln(20 - x) \\
&\Rightarrow (x+1)\ln 2 = \ln(20 - x) \\
&\Rightarrow x + 1 = \frac{\ln(20 - x)}{\ln 2} \\
&\Rightarrow \underline{\underline{x = \frac{\ln(20 - x)}{\ln 2} - 1}}.
\end{aligned}
$$

(b) Use the iterative formula (3)

$$x_{n+1} = \frac{\ln(20 - x_n)}{\ln 2} - 1, \; x_0 = 3,$$

to calculate the values of x_1, x_2, and x_3 to 3 decimal places.

Solution

$x_1 = 3.087\,462\,841$ (FCD) $=$ $\underline{\underline{3.087 \text{ (3 dp)}}}$

$x_2 = 3.080\,021\,199$ (FCD) $=$ $\underline{\underline{3.080 \text{ (3 dp)}}}$

$x_3 = 3.080\,655\,856$ (FCD) $=$ $\underline{\underline{3.081 \text{ (3 dp)}}}$

(c) Use your answer to part (b) to deduce the coordinates of the point A, giving your answers to one decimal place. (2)

Solution

Let $\mathrm{f}(x) = 2^{x+1} - 20 + x$. Now,
$\mathrm{f}(3.05) = -0.38\ldots$
$\mathrm{f}(3.15) = 0.90\ldots$
There is a (i) change of sign and (ii) continuity and we have a root in

$$3.05 < \alpha < 3.15,$$

i.e.,

$$\underline{\underline{(3.1, 13.9) \text{ (1 dp)}}}.$$

56. Figure 6 shows a sketch of part of the curve with equation

$$\mathrm{g}(x) = x^2(1 - x)\mathrm{e}^{-2x}, \; x \geqslant 0.$$

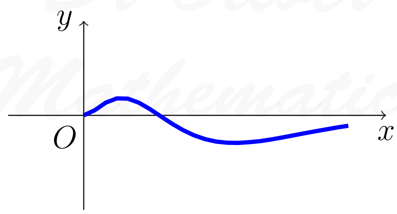


Figure 6: $g(x) = x^2(1-x)e^{-2x}$

(a) Show that $g'(x) = f(x)e^{-2x}$, where $f(x)$ is a cubic function to be found. (3)

Solution

$$\begin{aligned} g'(x) &= 2x(1-x)e^{-2x} - x^2e^{-2x} - 2x^2(1-x)e^{-2x} \\ &= e^{-2x}[2x(1-x) - x^2 - 2x^2(1-x)] \\ &= e^{-2x}(2x - 2x^2 - x^2 - 2x^2 + 2x^3) \\ &= \underline{\underline{e^{-2x}(2x^3 - 5x^2 + 2x)}}. \end{aligned}$$

(b) Hence find the range of g. (6)

Solution

$$\begin{aligned} g'(x) = 0 &\Rightarrow e^{-2x}(2x^3 - 5x^2 + 2x) = 0 \\ &\Rightarrow x(2x^2 - 5x + 2) = 0 \\ &\Rightarrow x(2x-1)(x-2) = 0 \\ &\Rightarrow x = 0,\ x = \tfrac{1}{2},\ \text{ or } x = 2. \end{aligned}$$

Now, $g(0) = 0$,
$g(\frac{1}{2}) = \frac{1}{8}e^{-1}$,
$g(2) = -4e^{-4}$.
Hence,

$$\underline{\underline{-4e^{-4} \leqslant g(x) \leqslant \tfrac{1}{8}e^{-1}}}.$$

(c) State a reason why the function $g^{-1}(x)$ does not exist. (1)

Solution

Because g is a <u>many-to-one function</u>.

57. Figure 7 shows a sketch of part of the curve with equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = |4\mathrm{e}^{2x} - 25|,\ x \in \mathbb{R}.$$

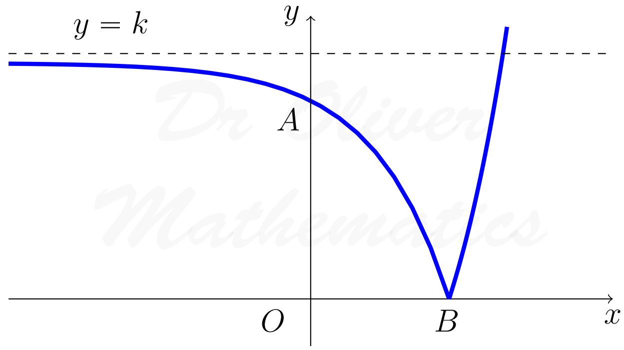


Figure 7: $\mathrm{g}(x) = |4\mathrm{e}^{2x} - 25|$

The curve cuts the y-axis at the point A and meets the x-axis at the point B. The curve has an asymptote $y = k$, where k is a constant.

(a) Find, giving each answer in its simplest form,

(i) the y-coordinate of the point A, (1)

Solution
$25 - 4 = \underline{\underline{21}}$.

(ii) the exact x-coordinate of the point B, (3)

Solution

$$4\mathrm{e}^{2x} - 25 = 0 \Rightarrow \mathrm{e}^{2x} = \tfrac{25}{4} \Rightarrow x = \tfrac{1}{2}\ln\tfrac{25}{4} \Rightarrow \underline{\underline{x = \ln\tfrac{5}{2}}}.$$

(iii) the value of the constant k. (1)

Solution
$\underline{\underline{25}}$.

The equation $\mathrm{g}(x) = 2x + 43$ has a positive root at $x = \alpha$.

(b) Show that α is a solution of $x = \frac{1}{2}\ln(\frac{1}{2}x + 17)$. (2)

Solution

$$\begin{aligned} 4e^{2x} - 25 = 2x + 43 &\Rightarrow 4e^{2x} = 2x + 68 \\ &\Rightarrow e^{2x} = \tfrac{1}{2}x + 17 \\ &\Rightarrow 2x = \ln(\tfrac{1}{2}x + 17) \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln(\tfrac{1}{2}x + 17)}}. \end{aligned}$$

The iteration formula

$$x_{n+1} = \tfrac{1}{2}\ln(\tfrac{1}{2}x_n + 17)$$

can be used to find an approximation for α.

(c) Taking $x_0 = 1.4$, find the values of x_1 and x_2. Give each answer to 4 decimal places. (2)

Solution

$x_1 = 1.436\,782\,32$ (FCD) $= \underline{\underline{1.4368 \text{ (4 dp)}}}$

$x_2 = 1.437\,301\,574$ (FCD) $= \underline{\underline{1.4373 \text{ (4 dp)}}}$

(d) By choosing a suitable interval, show that $\alpha = 1.437$ to 3 decimal places. (2)

Solution

Now, $h(x) = 4e^{2x} - 2x - 68$ and $g(1.4365) = -0.11\ldots$
$g(1.4375) = 0.026\ldots$
There is a (i) change of sign and (ii) continuity and we have a root in

$$1.4365 < \alpha < 1.4375,$$

i.e.,

$$\underline{\underline{1.437 \text{ (3 dp)}}}.$$

58. Find, using calculus, the x-coordinate of the turning point of the curve with equation (5)

$$y = e^{3x}\cos 4x,\ \tfrac{\pi}{4} \leqslant x < \tfrac{\pi}{2}.$$

Give your answer to 4 decimal places.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\mathrm{e}^{3x}\cos 4x - 4\mathrm{e}^{3x}\sin 4x = 0\\ &\Rightarrow 3\cos 4x - 4\sin 4x = 0\\ &\Rightarrow 3\cos 4x = 4\sin 4x\\ &\Rightarrow \tan 4x = \tfrac{3}{4}\\ &\Rightarrow 4x = \arctan\tfrac{3}{4},\ \arctan\tfrac{3}{4} + \pi\\ &\Rightarrow x = \tfrac{1}{4}(\arctan\tfrac{3}{4} + \pi)\\ &\Rightarrow x = 0.946\,273\,440\,6\ \text{(FCD)}\\ &\Rightarrow x = 0.9463\ \text{(4 dp)}\end{aligned}$$

59. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the equation

$$x = D\mathrm{e}^{-0.2t},$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams, and t is time time in hours after the antibiotic has been given. A first dose of 15 mg of the antibiotic is given.

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places. (2)

Solution

$$x = 15\mathrm{e}^{-0.8} = 6.739\,934\,462\ \text{(FCD)} = \underline{\underline{6.740\ \text{mg (3 dp)}}}.$$

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

(b) show that the total amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places. (2)

Solution

$$x = 15\mathrm{e}^{-1.4} + 15\mathrm{e}^{-0.4} = 13.753\,755\,515\ \text{(FCD)} = \underline{\underline{13.754\ \text{mg (3 dp)}}}.$$

No more doses of the antibiotic are given. At time T hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

(c) Show that $T = a\ln(b + \frac{b}{e})$, where a and b are integers to be determined. (4)

Solution

$$\begin{aligned}7.5 = 15e^{-0.2(T+5)} + 15e^{-0.2T} &\Rightarrow 7.5 = 15e^{-0.2T-1} + 15e^{-0.2T}\\ &\Rightarrow 7.5 = 15e^{-1}e^{-0.2T} + 15e^{-0.2T}\\ &\Rightarrow 7.5 = 15e^{-0.2T}(e^{-1}+1)\\ &\Rightarrow \tfrac{1}{2} = e^{-0.2T}(e^{-1}+1)\\ &\Rightarrow \frac{1}{2(e^{-1}+1)} = e^{-0.2T}\\ &\Rightarrow -0.2T = \ln\left(\frac{1}{2(e^{-1}+1)}\right)\\ &\Rightarrow T = -5\ln\left(\frac{1}{2(e^{-1}+1)}\right)\\ &\Rightarrow T = 5\ln[2(e^{-1}+1)]\\ &\Rightarrow \underline{\underline{T = 5\ln\left(2+\frac{2}{e}\right)}}.\end{aligned}$$

60. Find the exact solutions, in their simplest form, to the equations

(a) $e^{3x-9} = 8$, (3)

Solution

$$\begin{aligned}e^{3x-9} = 8 &\Rightarrow 3x - 9 = \ln 8\\ &\Rightarrow 3x = 9 + \ln 8\\ &\Rightarrow x = \tfrac{1}{3}(9 + \ln 8)\\ &\Rightarrow x = 3 + \tfrac{1}{3}\ln 8\\ &\Rightarrow x = 3 + \ln 8^{\frac{1}{3}}\\ &\Rightarrow \underline{\underline{x = 3 + \ln 2}}.\end{aligned}$$

(b) $\ln(2y + 5) = 2 + \ln(4 - y)$. (4)

Solution

$$\begin{aligned}\ln(2y+5) = 2 + \ln(4-y) &\Rightarrow \ln(2y+5) - \ln(4-y) = 2 \\ &\Rightarrow \ln\left(\frac{2y+5}{4-y}\right) = 2 \\ &\Rightarrow \frac{2y+5}{4-y} = \mathrm{e}^2 \\ &\Rightarrow 2y+5 = \mathrm{e}^2(4-y) \\ &\Rightarrow 2y+5 = 4\mathrm{e}^2 - y\mathrm{e}^2 \\ &\Rightarrow 2y + y\mathrm{e}^2 = 4\mathrm{e}^2 - 5 \\ &\Rightarrow y(2+\mathrm{e}^2) = 4\mathrm{e}^2 - 5 \\ &\Rightarrow \underline{\underline{y = \frac{4\mathrm{e}^2-5}{2+\mathrm{e}^2}}}.\end{aligned}$$

61. The number of rabbits on an island is modelled by the equation

$$P = \frac{100\mathrm{e}^{-0.1t}}{1+3\mathrm{e}^{-0.9t}} + 40,\ t \in \mathbb{R},\ t \geqslant 0,$$

where P is the number of rabbits, t years after they were introduced onto the island. A sketch of the graph of P against t is shown in Figure 8.

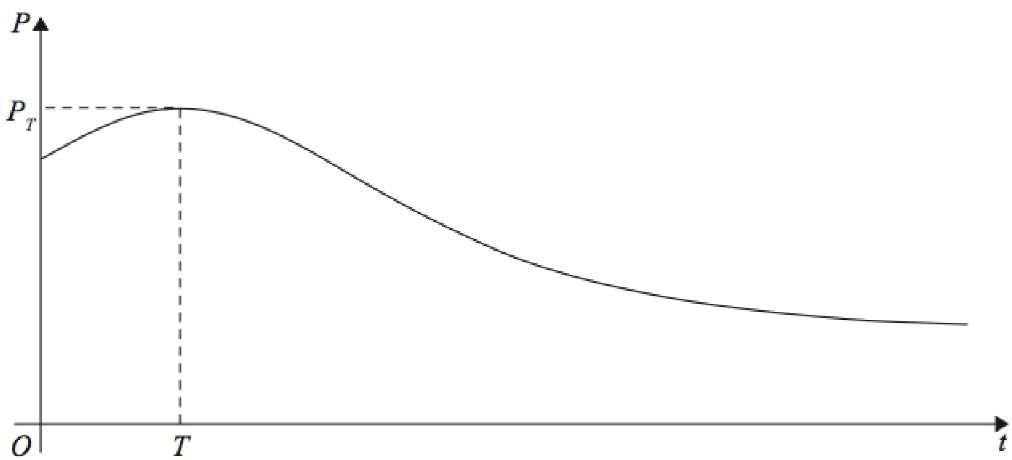


Figure 8: the number of rabbits on an island

(a) Calculate the number of rabbits that were introduced onto the island. (1)

Solution
$t = 0 \Rightarrow P = \frac{100}{1+3} + 40 = 25 + 40 = \underline{\underline{65}}$.

(b) Find $\dfrac{\mathrm{d}P}{\mathrm{d}t}$. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{(1+3\mathrm{e}^{-0.9t}) \times (-10\mathrm{e}^{-0.1t}) - 100\mathrm{e}^{-0.1t} \times (-2.7\mathrm{e}^{-0.9t})}{(1+3\mathrm{e}^{-0.9t})^2} \\ &= \frac{-10\mathrm{e}^{-0.1t} - 30\mathrm{e}^{-t} + 270\mathrm{e}^{-t}}{(1+3\mathrm{e}^{-0.9t})^2} \\ &= \underline{\underline{\frac{240\mathrm{e}^{-t} - 10\mathrm{e}^{-0.1t}}{(1+3\mathrm{e}^{-0.9t})^2}}}.\end{aligned}$$

The number of rabbits initially increases, reaching a maximum value P_T when $t = T$.

(c) Using your answer from part (b), calculate (4)

(i) the value of T to 2 decimal places,

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} = 0 &\Rightarrow 240\mathrm{e}^{-t} - 10\mathrm{e}^{-0.1t} = 0 \\ &\Rightarrow 24\mathrm{e}^{-t} = \mathrm{e}^{-0.1t} \\ &\Rightarrow \mathrm{e}^{0.9t} = 24 \\ &\Rightarrow 0.9t = \ln 24 \\ &\Rightarrow \underline{\underline{t = \tfrac{10}{9}\ln 24}}.\end{aligned}$$

(ii) the value of P_T to the nearest integer.

Solution

$$\begin{aligned}t = \tfrac{10}{9}\ln 24 &\Rightarrow P_T = 102.443\,993\,9 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{P_T = 102 \text{ (nearest integer)}}}.\end{aligned}$$

For $t > T$, the number of rabbits decreases, as shown in Figure 8, but never falls below k, where k is a positive constant.

(d) Use the model to state the maximum value of k. (1)

Solution

$\underline{\underline{P \to 40}}$ as $t \to \infty$.