Further Pure Mathematics 2: Part 1

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Further Mathematics

Inequalities

For inequalities involving rational algebraic expressions, either multiply each side by the square of the denominators or take all terms to one side and combine them to a single expression. For example, suppose

$$\frac{1}{x} > \frac{2}{x-1}.$$

You can either multiply each side by $x^2(x-1)^2$ to get

$$x(x-1)^2 > 2x^2(x-1)$$

or you can take all terms to one side and combine to get a single expression

$$\frac{1}{x} - \frac{2}{x-1} > 0 \Rightarrow \frac{(x-1)-2}{x(x-1)}.$$

Fully factorise the resulting expression to identify the critical values and then use your favourite method (table of signs, graph, etc) to identify the values of x that satisfy the inequality.

Series

You need to know, be able to prove (by induction, method of differences), and be able to apply the following results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$
$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Method of Differences

You need to be able to use the method of differences—and lay out a solution properly—using the principle that if we sum terms of the form $u_r = f(r) - f(r+k)$ then many of the terms of the sum will cancel, leaving us with a simple expression. For example, if $u_r = \frac{1}{r} - \frac{1}{r+1}$, then

$$\begin{aligned}&u_3 + u_4 + \ldots + u_{27}\\ =&\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\ldots+\left(\tfrac{1}{27}-\tfrac{1}{28}\right)\\ =&\tfrac{1}{3}-\tfrac{1}{28}.\end{aligned}$$

Complex Numbers

The *modulus-argument form* of a complex number $z = x + iy$ is

$$z = r(\cos\theta + \mathrm{i}\sin\theta),$$

where the *modulus*, r, is given by

$$r = \sqrt{x^2+y^2}$$

and θ is the *argument*; it is a *principal argument* if $-\pi < \theta \le \pi$. Some key results are

- $|z_1 z_2| = |z_1|\,|z_1|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_1|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

although note that we may not have a principal argument after the arguments are combined.

de Moivre's theorem

You need to be able to prove that

$$[r(\cos\theta + i\sin\theta)]^n \equiv r^n(\cos n\theta + i\sin n\theta)$$

for all $n \in \mathbb{Z}$ and use this to find the roots of complex numbers as well as writing powers of sine and cosine in terms of multiple angles and vice versa.

Loci in the complex plane

- $|z-(a+ib)| = c$: circle, centre $a+ib$, radius c;
- $|z-(a+ib)| = |z-(c+id)|$: the perpendicular bisector of the line joining $a+ib$ and $c+id$;
- $|z-(a+ib)| = \lambda\,|z-(c+id)|$: this is an circle if $\lambda \ne 1$;
- $\arg(z-(a+ib)) = \theta$: half-line, emerging from $a+ib$ at an angle θ to the real axis;
- $\arg\frac{z-(a+ib)}{z-(c+id)} = \theta$: arc of a circle (major if θ is acute, semicircle if θ is $\frac{\pi}{2}$, and minor if θ is obtuse), starting at $a+ib$ and moving in an anticlockwise sense to end at $c+id$;
- With inequalities, use a solid boundary for inclusive inequality, a dotted boundary for exclusive inequality, and (if required) shade the correct region.

Maclaurin Series

If $f(x)$ is a continuous function such that all of derivatives have a finite value when evaluated at zero, then the *Maclaurin series* for the function is

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2!}f''(0)x^2 + \tfrac{1}{3!}f'''(0)x^3 + \ldots$$

You should know the following Maclaurin series:

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots,\ x \in \mathbb{R}$
- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots,\ -1 < x \le 1$
- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots,\ x \in \mathbb{R}$
- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots,\ x \in \mathbb{R}$
- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots,\ -1 < x < 1$

Taylor Series

There are two versions of the *Taylor series* — both are derived from the Maclaurin series — and these are used for making approximations to the value of $f(x)$ close to $x = a$:

$$f(x+a) = f(a)+f'(a)x+\tfrac{1}{2!}f''(a)x^2+\tfrac{1}{3!}f'''(a)x^3+\ldots$$

and

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2!}f''(a)(x-a)^2 + \ldots$$

First Order Differential Equations

To solve a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we use the *integrating factor*

$$\text{IF} = e^{\int P(x)\,dx}$$

to turn the original equation into an exact differential equation.

$$\begin{aligned}e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y &= e^{\int P(x)\,dx}Q(x)\\ \frac{d}{dx}\left(e^{\int P(x)\,dx}y\right) &= e^{\int P(x)\,dx}Q(x)\\ e^{\int P(x)\,dx}y &= \int e^{\int P(x)\,dx}Q(x)\,dx\end{aligned}$$

Note that, on the left-hand side, we always end up with just yIF. The right-hand side looks complicated but, in practice, the examination board will choose functions $P(x)$ and $Q(x)$ that mean it can be integrated using your existing knowledge.

Second Order Differential Equations

Given the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c = f(x),$$

we form the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0$$

and solve it and use the two solutions to form the *complementary function*, CF:

- if there are two real distinct roots α and β then the CF is $y = Ae^{\alpha x} + Be^{\beta x}$ for some constants A and B;
- if there is one real repeated root α then the CF is $y = (A + Bx)e^{\alpha x}$;
- if the roots are a complex conjugate pair $\alpha \pm i\beta$ then the CF is $y = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$.

For the *particular integral*, PI,

- if $f(x)$ is a polynomial then use the most general polynomial of the same degree;
- if $f(x)$ is an exponential then use an exponential (unless, as explained below, the exponential is part of the CF);
- if $f(x)$ is sine or cosine or both then use both trigonometric functions.

If your first guess at the PI includes a term that is also part of the CF then multiply your initial guess by x and try that. You can now find the *general solution* to the equation by adding the complementary function and the particular integral, i.e.,

$$\text{GS} = \text{CF} + \text{PI}.$$

You need to be able to use a substitution if one is given to you. The most popular substitution is $x = e^u$ and you should ensure that you can derive the results

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}$$

and

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{\mathrm{d}y}{\mathrm{d}u}$$

both accurately and quickly (but you need to be able to work with a range of substitutions).