# Dr Oliver Mathematics
# Cambridge O Level Additional Mathematics
# 2008 November Paper 2: Calculator
# 2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Given that (4)

$$\mathbf{A} = \begin{pmatrix} 13 & 6 \\ 7 & 4 \end{pmatrix},$$

find the inverse matrix $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$\begin{aligned} 13x + 6y &= 41 \\ 7x + 4y &= 24. \end{aligned}$$

**Solution**

Well,

$$\begin{aligned} \det \mathbf{A} &= (13 \times 4) - (6 \times 7) \\ &= 52 - 42 \\ &= 10 \end{aligned}$$

so

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{10}\begin{pmatrix} 4 & -6 \\ -7 & 13 \end{pmatrix}}}.$$

In matrix notation,

$$\begin{aligned} & \begin{pmatrix} 13 & 6 \\ 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 41 \\ 24 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 4 & -6 \\ -7 & 13 \end{pmatrix}\begin{pmatrix} 41 \\ 24 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 20 \\ 25 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2.5 \end{pmatrix}; \end{aligned}$$

so,

$$\underline{\underline{x = 2,\ y = 2.5}}.$$

2. Variables $x$ and $y$ are connected by the equation (4)

$$y = (2x - 9)^3.$$

Given that $x$ is increasing at the rate of 4 units per second, find the rate of increase of $y$ when $x = 7$.

**Solution**

Well,

$$y = (2x - 9)^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6(2x - 9)^2$$

and

$$x = 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6(2 \times 7 - 9)^2$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 150.$$

Finally,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \Rightarrow 150 = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{4}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \underline{\underline{600}}.$$

3. Find the set of values of $m$ for which the line (5)

$$y = mx + 2$$

does not meet the curve

$$y = x^2 - 5x + 18.$$

**Solution**

Well,

$$x^2 - 5x + 18 = mx + 2 \Rightarrow x^2 - (5+m)x + 16 = 0$$

but

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow (5+m)^2 - 4 \times 1 \times 16 < 0 \\ &\Rightarrow (5+m)^2 - 64 < 0 \end{aligned}$$

difference of two squares:

$$\begin{aligned} &\Rightarrow [(5+m) - 8][(5+m) + 8] < 0 \\ &\Rightarrow (m-3)(m+13) < 0. \end{aligned}$$

We need a 'table of signs':

| | $m < -13$ | $m = -13$ | $-13 < m < 3$ | $m = 3$ | $m > 3$ |
|---|---|---|---|---|---|
| $m + 13$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $m - 3$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $(m+13)(m-3)$ | $+$ | 0 | $-$ | 0 | $+$ |

Hence,

$$\underline{\underline{-13 < m < 3}}.$$

4. (a) Differentiate (2)

$$x \ln x$$

with respect to $x$.

**Solution**

Well,

$$\begin{aligned} u = x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ v = \ln x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x} \end{aligned}$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x)\left(\frac{1}{x}\right) + (1)(\ln x)$$
$$= \underline{\underline{1 + \ln x}}.$$

(b) Hence find (3)

$$\int \ln x \,\mathrm{d}x.$$

**Solution**

$$\int \ln x \,\mathrm{d}x = \int (\ln x + 1 - 1)\,\mathrm{d}x$$
$$= \int \left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right)\,\mathrm{d}x$$
$$= \underline{\underline{x \ln x - x + c}}.$$

5. Solve the equation

(a) $\dfrac{4^x}{2^{5-x}} = \dfrac{2^{4x}}{8^{x-3}}$. (3)

**Solution**

$$\frac{4^x}{2^{5-x}} = \frac{2^{4x}}{8^{x-3}} \Rightarrow \frac{(2^2)^x}{2^{5-x}} = \frac{2^{4x}}{(2^3)^{x-3}}$$
$$\Rightarrow \frac{2^{2x}}{2^{5-x}} = \frac{2^{4x}}{2^{3(x-3)}}$$
$$\Rightarrow 2^{2x-(5-x)} = 2^{4x-3(x-3)}$$
$$\Rightarrow 2^{3x-5} = 2^{x+9}$$

equate the indices:

$$\Rightarrow 3x - 5 = x + 9$$
$$\Rightarrow 2x = 14$$
$$\Rightarrow \underline{\underline{x = 7}}.$$

(b) $\log_{10}(2y + 10) + \log_{10} y = 2.$ (3)

**Solution**

$$\begin{aligned}\log_{10}(2y+10) + \log_{10} y = 2 &\Rightarrow \log_{10}[y(2y+10)] = 2\\ &\Rightarrow y(2y+10) = 10^2\\ &\Rightarrow 2y^2 + 10y = 100\\ &\Rightarrow 2y^2 + 10y - 100 = 0\\ &\Rightarrow 2(y^2 + 5y - 50) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +5\\ \text{multiply to:} & -50\end{array}\right\} +10,\ -5$$

$$\begin{aligned}&\Rightarrow 2(y+10)(y-5) = 0\\ &\Rightarrow y = -10 \text{ or } y = 5;\end{aligned}$$

but $y \neq -10$ (why?) so $\underline{\underline{y = 5}}$.

6. (a) A sports team of 3 attackers, 2 centres and 4 defenders is to be chosen from a squad of 5 attackers, 3 centres and 6 defenders. (3)

Calculate the number of different ways in which this can be done.

**Solution**

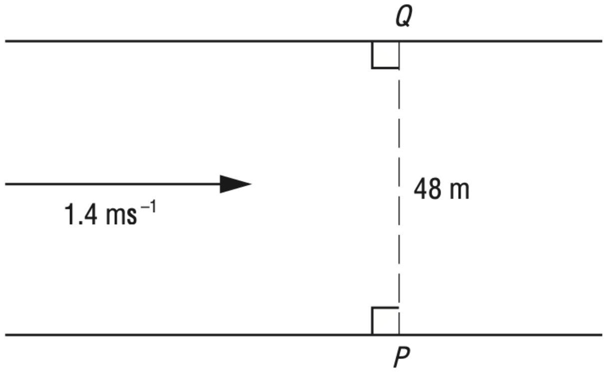
$$\begin{aligned}\binom{5}{3} \times \binom{3}{2} \times \binom{6}{4} &= 10 \times 3 \times 15\\ &= \underline{\underline{450 \text{ ways}}}.\end{aligned}$$

(b) How many different 4-digit numbers greater than 3 000 can be formed using the six digits 1, 2, 3, 4, 5, and 6 if no digit can be used more than once? (3)

**Solution**

$$\begin{aligned}\binom{4}{1} \times 5 \times 4 \times 3 &= 4 \times 60\\ &= \underline{\underline{240 \text{ ways}}}.\end{aligned}$$

7. The diagram shows a river with parallel banks.




The river is 48 m wide and is flowing with a speed of 1.4 ms$^{-1}$.

A boost travels in a straight line from a point $P$ on one bank to a point $Q$ which is on the other bank directly opposite $P$.

Given that the boat takes 10 seconds to cross the river, find

(a) the speed of the boat in still water, (4)

**Solution**

Well, the speed of travel is

$$\frac{48}{10} = 4.8 \text{ ms}^{-1}$$

and, to finish, we need Pythagoras' theorem:

$$\begin{aligned}\text{speed of the boat in still water} &= \sqrt{1.4^2 + 4.8^2} \\ &= \sqrt{1.96 + 23.04} \\ &= \sqrt{25} \\ &= \underline{\underline{5 \text{ ms}^{-1}}}.\end{aligned}$$

(b) the angle to the bank at which the boat should be steered. (2)

**Solution**

Well,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\alpha = \frac{4.8}{1.4}$$
$$\Rightarrow \tan\alpha = \tfrac{24}{7}$$
$$\Rightarrow \alpha = 73.739\,795\,29 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\alpha = 73.7^\circ \text{ (3 sf)}}}.$$

8. The function f is defined, for $0 \leqslant x \leqslant 2\pi$, by

$$\mathrm{f}(x) = 3 + 5\sin 2x.$$

State

(a) the amplitude of f, (1)

**Solution**
$\underline{\underline{5}}$.

(b) the period of f, (1)

**Solution**

$$\frac{2\pi}{2} = \underline{\underline{\pi}}.$$

(c) the maximum and minimum values of f. (2)

**Solution**
The maximum value is

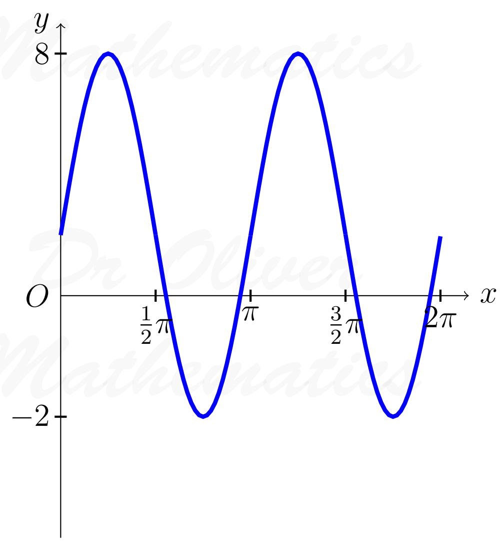
$$3 + 5 = \underline{\underline{8}}$$

and the minimum value is

$$3 - 5 = \underline{\underline{-2}}.$$

(d) Sketch the graph of $y = \mathrm{f}(x)$. (3)

**Solution**




9. The line (8)

$$y = 2x - 9$$

intersects the curve

$$x^2 + y^2 + xy + 3x = 46$$

at the points $A$ and $B$.

Find the equation of the perpendicular bisector of $AB$.

**Solution**

$$x^2 + y^2 + xy + 3x = 46$$
$$\Rightarrow \quad x^2 + (2x-9)^2 + x(2x-9) + 3x = 46$$

| $\times$ | $2x$ | $-9$ |
|---|---|---|
| $2x$ | $4x^2$ | $-18x$ |
| $-9$ | $-18x$ | $+81$ |

$$\Rightarrow \quad x^2 + (4x^2 - 36x + 81) + (2x^2 - 9x) + 3x = 46$$
$$\Rightarrow \quad 7x^2 - 42x + 35 = 0$$
$$\Rightarrow \quad 7(x^2 - 6x + 5) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -6\\ \text{multiply to:} & +5\end{array}\right\}-5,\ -1$$

$$\begin{aligned}&\Rightarrow \quad 7(x-5)(x-1)=0\\ &\Rightarrow \quad x=1 \text{ or } x=5\\ &\Rightarrow \quad y=-7 \text{ or } y=1;\end{aligned}$$

so, $A(1,-7)$ and $B(5,1)$ (or vice versa).

Now, the midpoint, which we will label as $C$, is

$$\left(\frac{1+5}{2},\frac{-7+1}{2}\right)=(3,-3).$$

Next,

$$\begin{aligned}m_{AB}&=\frac{1-(-7)}{5-1}\\ &=\frac{8}{4}\\ &=2,\end{aligned}$$

and

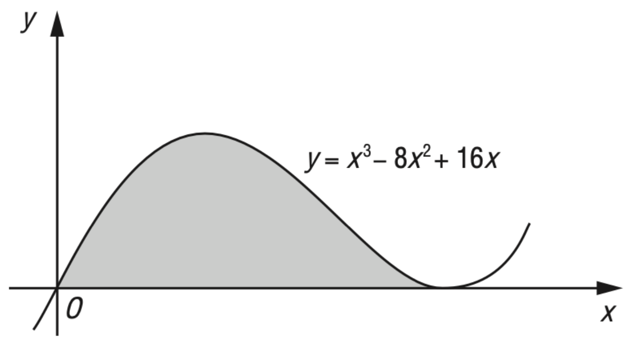
$$m_{\text{tangent}}=-\tfrac{1}{2}.$$

Finally, the equation of the perpendicular bisector of $AB$ is

$$\begin{aligned}y+3=-\tfrac{1}{2}(x-3)&\Rightarrow y+3=-\tfrac{1}{2}x+\tfrac{3}{2}\\ &\Rightarrow \underline{\underline{y=-\tfrac{1}{2}x-\tfrac{3}{2}}}.\end{aligned}$$

10. The diagram shows part of the curve

$$y=x^3-8x^2+16x.$$




(a) Show that the curve has a minimum point at $(4, 0)$ and find the coordinates of the maximum point. (4)

**Solution**

Well,

$$y = x^3 - 8x^2 + 16x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 16x + 16$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^2 - 16x + 16 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -16 \\ \text{multiply to:} & (+3) \times (+16) = +48\end{array}\right\} -12, -4$$

e.g.,

$$\begin{aligned}&\Rightarrow 3x^2 - 12x - 4x + 16 = 0 \\ &\Rightarrow 3x(x-4) - 4(x-4) = 0 \\ &\Rightarrow (3x-4)(x-4) = 0 \\ &\Rightarrow x = 1\tfrac{1}{3} \text{ or } x = 4 \\ &\Rightarrow y = 9\tfrac{13}{27} \text{ or } y = 0;\end{aligned}$$

so, the curve has a minimum point at $\underline{\underline{(4, 0)}}$ and the maximum point is $\underline{\underline{(1\frac{1}{3}, 9\frac{13}{27})}}$.

(b) Find the area of the shaded region enclosed by the $x$-axis and the curve. (4)

**Solution**

$$\begin{aligned}\text{Area} &= \int_0^4 (x^3 - 8x^2 + 16x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 8x^2\right]_{x=0}^4 \\ &= \left(64 - 170\tfrac{2}{3} + 128\right) - (0 - 0 + 0) \\ &= \underline{\underline{21\tfrac{1}{3}}}.\end{aligned}$$

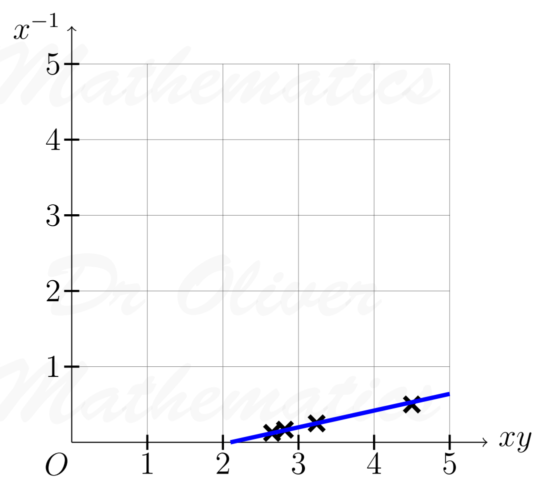
11. The table shows experimental values of two variables $x$ and $y$.

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $y$ | 2.25 | 0.81 | 0.47 | 0.33 |

(a) Using graph paper, plot $xy$ against $\dfrac{1}{x}$ and draw a straight line graph. (3)

**Solution**

| $xy$ | 4.5 | 3.24 | 2.82 | 2.65 |
|---|---|---|---|---|
| $x^{-1}$ | 0.5 | 0.25 | 0.167 | 0.125 |




(b) Use your graph to express $y$ in terms of $x$. (5)

**Solution**

The line goes through $(2.1, 0)$ and $(5, 0.64)$:

$$m = \frac{0.64 - 0}{5 - 2.1} = \tfrac{32}{145}$$

and the equation is

$$x^{-1} - 0 = \tfrac{32}{145}(xy - 2.1) \Rightarrow \frac{145}{32x} = xy - 2.1$$
$$\Rightarrow \frac{145}{32x} + 2.1 = xy$$
$$\Rightarrow \underline{\underline{y = \frac{145}{32x^2} + \frac{21}{10x}}}.$$

(c) Estimate the value of $x$ and of $y$ for which (3)

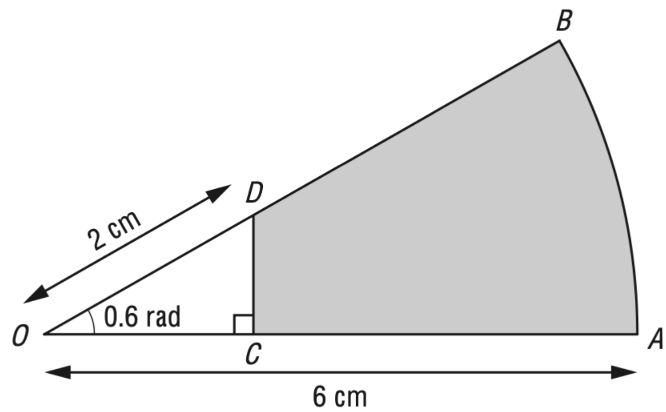
$$xy = 4.$$

**Solution**

$$xy = 4 \Rightarrow x^{-1} = 0.43$$
$$\Rightarrow \underline{\underline{x = 2\tfrac{14}{43}}}$$
$$\Rightarrow \underline{\underline{y = 1\tfrac{18}{25}}}.$$

**EITHER**

12. The diagram shows a sector $AOB$ of a circle with centre $O$ and radius 6 cm.




- Angle $AOB = 0.6$ radians.
- The point $D$ lies on $OB$ such that the length of $OD$ is 2 cm.
- The point $C$ lies on $OA$ such that $OCD$ is a right angle.

(a) Show that the length of $OC$ is approximately 1.65 cm and find the length of $CD$. (4)

**Solution**

Well,

$$\begin{aligned}\cos = \frac{\text{adj}}{\text{hyp}} &\Rightarrow \cos 0.6 = \frac{OC}{2}\\ &\Rightarrow OC = 2\cos 0.6\\ &\Rightarrow OC = 1.650\,671\,23 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{OC = 1.65 \text{ cm (3 sf)}}},\end{aligned}$$

as required. Now,

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 0.6 = \frac{CD}{2}\\ &\Rightarrow CD = 2\sin 0.6\\ &\Rightarrow CD = 1.129\,284\,947 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{CD = 1.13 \text{ cm (3 sf)}}}.\end{aligned}$$

(b) Find the perimeter of the shaded region. (3)

**Solution**

Now,

$$\begin{aligned}\text{perimeter} &= AC + CD + BD + \text{arc } AB \\ &= (6 - 2\cos 0.6) + 2\sin 0.6 + 4 + (6 \times 0.6) \\ &= 13.078\,613\,72 \text{ (FCD)} \\ &= \underline{\underline{13.1 \text{ cm (3 sf)}}}.\end{aligned}$$

(c) Find the area of the shaded region. (3)

**Solution**

$$\begin{aligned}\text{Area} &= \text{whole area} - \text{triangle} \\ &= (\tfrac{1}{2} \times 0.6 \times 6^2) - (\tfrac{1}{2} \times 2\sin 0.6 \times 2\cos 0.6) \\ &= 9.867\,960\,914 \text{ (FCD)} \\ &= \underline{\underline{9.87 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

**OR**

13. A particle moves in a straight line so that $t$ seconds after passing a fixed point $O$ its acceleration, $a$ ms$^{-2}$, is given by

$$a = 4t - 12.$$

Given that its speed at $O$ is 16 ms$^{-1}$, find

(a) the values of $t$ at which the particle is stationary, (5)

**Solution**

Well,

$$a = 4t - 12 \Rightarrow v = 2t^2 - 12t + c,$$

for some constant $c$. Now,

$$\begin{aligned}t = 0, v = 16 &\Rightarrow 0 + 0 + c = 16 \\ &\Rightarrow c = 16,\end{aligned}$$

and so

$$v = 2t^2 - 12t + 16.$$

Next,

$$\begin{aligned} v = 0 &\Rightarrow 2t^2 - 12t + 16 = 0 \\ &\Rightarrow 2(t^2 - 6t + 8) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -6 \\ \text{multiply to:} & +8 \end{array}\right\} -4, -2$$

$$\begin{aligned} &\Rightarrow 2(t-4)(t-2) = 0 \\ &\Rightarrow \underline{\underline{t = 2 \text{ or } t = 4}}. \end{aligned}$$

(b) the distance the particle travels in the fifth second. (5)

**Solution**

Well,

$$v = 2t^2 - 12t + 16 \Rightarrow s = \tfrac{2}{3}t^3 - 6t^2 + 16t + d,$$

for some constant $d$. Now,

$$\begin{aligned} \text{fifth second} &= s(5) - s(4) \\ &= (83\tfrac{1}{3} - 150 + 80 + d) - (42\tfrac{2}{3} - 96 + 64 + d) \\ &= \underline{\underline{2\tfrac{2}{3} \text{ m}}}. \end{aligned}$$